The physics of gravitational waves

Selected derivations: generation, energy, ringdown, detection

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Follows arXiv:2303.11713 (PoS 440, 002) — §II.C, II.D, II.E, III.C, V, VII, VIII
Equation numbering and notation as in the lecture notes.

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ROADMAP

Five derivations, one thread

§ II.C–E
How waves propagate and are generated

Split the metric by its behaviour under rotations; find which parts actually propagate; then solve for the radiation of a source.

Result: **the quadrupole formula**

§ III.C
Why that derivation was not legitimate

Linear theory assumes weak gravity and flat-space conservation. Neither holds for a compact binary. Redo it exactly.

Result: **the same formula, now justified**

§ V
How much energy a wave carries

The equivalence principle forbids a local energy density for gravity. Yet waves clearly carry energy.

Result: **a non-local stress-energy**

§ VII
What rings after the merger

Perturb a black hole and impose the physical boundary conditions. The spectrum comes out discrete and complex.

Result: **quasi-normal modes**

§ VIII
What a detector actually measures

In TT gauge the mirrors do not move. Resolve the paradox, then drop the long-wavelength assumption.

Result: **the transfer function**

Split the perturbation by how it rotates

1. Write $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu}$ and classify each component under **spatial rotations**.

   Under rotations h_{tt} is a scalar, h_{ti} a vector, h_{ij} a tensor.

2. Helmholtz-decompose the vector and the tensor further.

   A vector splits into a gradient plus a curl; a tensor into two scalars, a divergenceless vector and a transverse traceless piece.

COUNTING

4 scalars (φ, γ, H, λ) + **2 divergenceless vectors** (β_i, ε_i, 2 each) + **1 TT tensor** (2) = **10**, as it must be for a symmetric 4×4 matrix.

$$
\begin{aligned}
h_{tt} &= 2\phi \\
h_{ti} &= \partial_i \gamma + \beta_i, \qquad \partial_i \beta^i = 0 \\
h_{ij} &= \tfrac{1}{3} H \delta_{ij} + \partial_{(i}\varepsilon_{j)} + \left(\partial_i \partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda + h^{\mathrm{TT}}_{ij}
\end{aligned}
$$

$$\partial_i \varepsilon^i = 0, \qquad \partial_i h^{\mathrm{TT}\,ij} = 0 = h^{\mathrm{TT}\,i}{}_i$$

UNIQUENESS

The split is invertible, so it carries no arbitrariness:

$$\gamma = \nabla^{-2}(\partial_i h_{ti}), \qquad \beta_i = h_{ti} - \partial_i\left[\nabla^{-2}(\partial_j h_{tj})\right]$$

Which pieces are pure gauge?

Decompose the generator of an infinitesimal coordinate change, $\xi^t = A$ and $\xi^i = \partial_i C + B_i$, and feed it into the transformation law:

$$\begin{aligned} \tilde{\phi} &= \phi - \partial_t A & \tilde{H} &= H - 2\nabla^2 C \\ \tilde{\beta}_i &= \beta_i - \partial_t B_i & \tilde{\lambda} &= \lambda - 2C \\ \tilde{\gamma} &= \gamma - A - \partial_t C & \tilde{\varepsilon}_i &= \varepsilon_i - 2B_i \end{aligned}$$

THE POINT

$\tilde{h}_{ij}^{\mathrm{TT}} = h_{ij}^{\mathrm{TT}}$ — the transverse traceless part is **gauge invariant** on its own.

1. Spend A, C, B_i to kill **γ, λ, ε_i**. What is left is the **Poisson gauge**:

$$h_{tt} = 2\phi, \qquad h_{ti} = \beta_i, \qquad h_{ij} = \tfrac{1}{3}H\delta_{ij} + h_{ij}^{\mathrm{TT}}$$

Unlike Lorenz gauge, this fixes the coordinates *completely* at linear order — no residual freedom.

2. Or build gauge-invariant combinations directly — **Bardeen's variables**:

$$\begin{aligned} \psi &= -\phi + \partial_t\gamma - \tfrac{1}{2}\partial_t^2\lambda \\ \theta &= \tfrac{1}{3}\left(H - \nabla^2\lambda\right) \\ \Sigma_i &= \beta_i - \tfrac{1}{2}\partial_t\varepsilon_i \end{aligned}$$

In Poisson gauge these reduce to −φ, H/3 and β_i: the two routes are equivalent.

Only two of the ten degrees of freedom propagate

Writing the Einstein tensor in Bardeen variables, decomposing the source the same way, and using $\partial_\mu T^{\mu\nu} = 0$ to eliminate the three combinations that the Bianchi identities make redundant, six independent equations survive:

$$\begin{aligned}
\nabla^2\theta &= -8\pi\rho && \text{(1 d.o.f.)}\\
\nabla^2\psi &= 4\pi\big(\rho + 3p - 3\partial_t S\big) && \text{(1 d.o.f.)}\\
\nabla^2\Sigma_i &= -16\pi S_i && \text{(2 d.o.f.)}\\
\Box\, h_{ij}^{\mathrm{TT}} &= -16\pi\sigma_{ij} && \text{(2 d.o.f.)}
\end{aligned}$$

READ THE OPERATORS

The first three are **elliptic** (Poisson-like): they are constraints, fixed instantaneously by the matter distribution. No propagation, no radiation.

THE WAVE

Only $h^{\mathrm{TT}}{}_{ij}$ obeys a **hyperbolic** equation. Gravitational waves are exactly the transverse traceless, gauge-invariant part of the metric — two polarisations.

The non-propagating modes are just Newton

1. Invert the Laplacian with its Green function, $\nabla^2 \frac{1}{|\vec{x}|} = -4\pi\, \delta^{(3)}(\vec{x})$
2. Far from a localised source, $\theta(t, \vec{x}) \approx \frac{2}{r} \int \rho\, \mathrm{d}^3 x' = \frac{2M}{r}$

   so it is natural to call $M = \int \rho\, \mathrm{d}^3 x$ the mass.
3. Check that M is **conserved**, using the decomposed conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \rho\, \mathrm{d}^3 x = \int \nabla^2 S\, \mathrm{d}^3 x = \oint \vec{\nabla} S \cdot \vec{n}\, \mathrm{d}^2 S = 0$$

Gauss' theorem, plus no matter at infinity. The same steps give $\bar{M} = M$ for ψ, and conservation of the momentum π_i for Σ_i.

$$\psi \sim \frac{M}{r}, \qquad \theta \sim \frac{M}{r}, \qquad \Sigma_i \sim \frac{\pi_i}{r}$$

INTERPRETATION

These three are not radiation — they are the **Newtonian potential** (ψ) and its relativistic companions: periastron precession and light bending (θ), frame dragging (Σ_i).

Beyond GR: these static scalar and vector sectors generally become *dynamical*, which is exactly why extra polarisations appear in modified gravity. Come back to this in § II.E.

Solving the wave equation for a source

$$\Box\, h_{ij}^{\mathrm{TT}} = -16\pi\, \sigma_{ij}$$

1. Retarded Green function of the flat d'Alembertian:

$$G(t,\vec{x}) = -\frac{1}{4\pi|\vec{x}|}\,\delta(t-|\vec{x}|), \qquad \Box G = \delta(t)\,\delta^{(3)}(\vec{x})$$

2. Hence the retarded solution

$$h_{ij}^{\mathrm{TT}}(t,\vec{x}) = 4\int \frac{\sigma_{ij}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'$$

3. But the source is $\boldsymbol{\sigma_{ij}}$, not T_{ij}. Project it out:

$$P_{ij} = \delta_{ij} - \nabla^{-2}\partial_i\partial_j \xrightarrow[\text{far field}]{} \delta_{ij} - n_i n_j$$

$$\sigma_{ij} = \left(P_i{}^k P_j{}^l - \tfrac{1}{2}P_{ij}P^{kl}\right)T_{kl} \equiv \mathcal{P}_{ij}{}^{kl}\,T_{kl}$$

4. Projectors and ∂_μ commute with $\Box$, so pull them out of the integral, then take $r \to \infty$:

$$h_{ij}^{\mathrm{TT}} = \frac{4}{r}\,\mathcal{P}_{ij}{}^{kl}\int T_{kl}(t-|\vec{x}-\vec{x}'|,\vec{x}')\,\mathrm{d}^3x'$$

Call this the **“Green formula”**. Remember it — it is about to cause trouble.

From the source integral to the quadrupole

1. Flat-space conservation $\partial_\mu T^{\mu\nu} = 0$ gives an **identity** that trades T^{ij} for second time derivatives of T^{tt}:

$$\partial_t^2\left(T^{tt}x^i x^j\right) = 2T^{ij} + \partial_k\partial_l\left(T^{kl}x^i x^j\right) - 2\partial_k\left(T^{ik}x^j + T^{kj}x^i\right)$$

2. The last two terms are total divergences — they integrate to surface terms and vanish for a **confined** source:

$$h_{ij}^{\mathrm{TT}} = \frac{2}{r}\,\mathcal{P}_{ij}{}^{kl}\int \partial_t^2\left(T^{tt}x'^k x'^l\right)\,\mathrm{d}^3x'$$

3. Introduce the inertia and quadrupole tensors:

$$I_{ij} = \int \mathrm{d}^3x'\,\rho\,x'^i x'^j, \qquad Q_{ij} = I_{ij} - \tfrac{1}{3}I\,\delta_{ij}$$

$$\boxed{h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 r}\,\mathcal{P}_{ij}{}^{kl}\,\ddot{Q}_{kl}}$$

Two assumptions that a binary violates

ASSUMPTION I

Linear perturbation theory, $h_{\mu\nu} \ll 1$.
Near a black hole or a neutron star the spacetime is simply *not* a small perturbation of Minkowski.

ASSUMPTION II

$\partial_\mu T^{\mu\nu} = 0$ on flat space.
But in flat space that *is* the geodesic equation — and its solutions are **straight lines**. A quasi-circular binary is not a straight line.

SO WHY DID IT WORK?

It does work — but the derivation cannot tell us that. We need the exact theory to know *which* of the two formulae above survives.

Exercise 2. Take an equal-mass Keplerian binary on a circular orbit in the (x, y) plane, and an observer far away along z. Compute the signal from the **Green formula**, then from the **quadrupole formula**.

The two amplitudes differ by a factor **2**.

THE DIAGNOSIS

Two expressions derived from the same equation, in the same gauge, disagree. At least one of them is being applied outside its domain of validity. Section III.C decides which.

Why the quadrupole is the leading term

Characterise the source by its multipole moments,

$$M = \int \rho \, \mathrm{d}^3x, \quad D^i = \int \rho \, x^i \, \mathrm{d}^3x, \quad L_i = \int \rho \, \epsilon_{ijk} x^j v^k \, \mathrm{d}^3x$$

and build a dimensionless *h* out of each. Since h must be dimensionless, each moment comes with a fixed power of c:

$$h \sim \frac{GM}{rc^2}$$

Monopole
M is conserved → static.

$$h \sim \frac{G\dot{D}}{rc^3} \sim \frac{GP}{rc^3}$$

Dipole
Ḋ = P is conserved → zero.

$$h \sim \frac{G\dot{L}}{rc^4}$$

Angular momentum
L is conserved → zero.

$$h \sim \frac{G\ddot{Q}}{rc^4}$$

Quadrupole
Nothing forbids Q̈ → **this radiates**.

BEYOND GR

Monopole, dipole and angular-momentum radiation **can** appear, because mass, momentum and angular momentum of matter need not be separately conserved once energy can be exchanged with extra gravitational degrees of freedom. This is precisely the sector that was static in § II.C.

Redo it without linearising

$$\boxed{\Box_\eta \bar{H}^{\mu\nu} = -16\pi\,\tau^{\mu\nu}}$$

1. Adopt **harmonic coordinates**. The x^α are scalars, not vectors, so

$$\Box\, x^\alpha = 0 \quad \Longleftrightarrow \quad \partial_\mu \bar{H}^{\mu\nu} = 0$$

2. expressed through the **pseudo-tensor**

$$\bar{H}^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g}\, g^{\mu\nu} \;=\; \bar{h}^{\mu\nu} + \mathcal{O}(h^2)$$

At linear order this is the trace-reversed perturbation, and harmonic gauge becomes Lorenz gauge.

NOTE

Nothing has been expanded yet. What follows is **exact**.

$$\tau^{\mu\nu} = (-g)\, T^{\mu\nu} + \frac{\Lambda^{\mu\nu}}{16\pi}$$

$$\Lambda^{\mu\nu} = 16\pi(-g)\, t^{\mu\nu}_{\mathrm{LL}} + \left(\partial_\beta \bar{H}^{\mu\alpha}\partial_\alpha \bar{H}^{\nu\beta} - \partial_\alpha\partial_\beta \bar{H}^{\mu\nu}\bar{H}^{\alpha\beta}\right)$$

WHAT CHANGED

The left side is the **flat** wave operator; everything non-linear has been moved into the source $\tau^{\mu\nu}$, which now contains the Landau–Lifshitz pseudo-tensor: the energy of the gravitational field itself gravitates.

Now the conservation law is the exact equation of motion

Take the divergence of the relaxed equations and use the gauge condition:

$$\partial_\mu \tau^{\mu\nu} = 0 \qquad \text{(exact equations of motion of matter)}$$

WHY THIS FIXES ASSUMPTION II

This is **not** flat-space conservation. It does not imply straight-line motion, because $\Lambda^{\mu\nu}$ contains **second derivatives** of $\bar{H}^{\mu\nu}$ — wavefronts do not follow straight lines even though the operator on the left is the flat one.

Repeat the § II.D steps verbatim, with τ in place of T:

$$\bar{H}^{ij}(t,\vec{x}) \approx \frac{4}{r}\int \tau^{ij}\Big(t-\frac{r}{c},\vec{x}'\Big)\,\mathrm{d}^3x'$$

$$\bar{H}^{ij} \approx \frac{2}{r}\,\ddot{Q}_{ij}, \qquad Q_{ij} = \int \tau^{tt} x'^i x'^j\,\mathrm{d}^3x'$$

Formally identical. The physics is in how τ relates to T.

$$\begin{aligned}
\tau^{tt} &= T^{tt}\Big[1+\mathcal{O}(c^{-2})\Big]\\
\tau^{ti} &= T^{ti}\Big[1+\mathcal{O}(c^{-2})\Big]\\
\tau^{ij} &= \Big[T^{ij} + \frac{1}{4\pi}\Big(\partial^i\phi\,\partial^j\phi - \tfrac{1}{2}\delta^{ij}\partial_k\phi\,\partial^k\phi\Big)\Big]\Big[1+\mathcal{O}(c^{-2})\Big]
\end{aligned}$$

Only one of the two formulae survives

THE QUADRUPOLE FORMULA

At leading PN order $\tau^{tt} \approx T^{tt}$, so the quadrupole moment built from τ coincides with the one built from ρ. The formula derived in linear theory is **correct**.

THE GREEN FORMULA

But $\tau^{ij} \neq T^{ij}$ at the same order: the field-energy term $\partial^i\varphi\partial^j\varphi$ is *not* subleading. The linear-theory Green formula is **wrong** when applied to a binary.

RESOLUTION OF EXERCISE 2

The missing $\partial\varphi\partial\varphi$ terms are exactly what supplies the factor of **2**. Gravitational binding energy is part of the source.

Exercise 3. Show that the extra terms in τ^{ij} resolve the factor-2 discrepancy.

Hint: in the sense of distributions, $\nabla^2 g(\mathbf{x}, \mathbf{y}', \mathbf{y}'') = |\mathbf{x}-\mathbf{y}'|^{-1}|\mathbf{x}-\mathbf{y}''|^{-1}$ is solved by $g = \ln(|\mathbf{x}-\mathbf{y}'| + |\mathbf{x}-\mathbf{y}''| + |\mathbf{y}'-\mathbf{y}''|) + \text{const.}$

MORAL

A formula being *derivable* in linear theory says nothing about its validity for strong-field sources. Only the exact treatment tells you which linear-theory results you are allowed to keep.

Gravity has no local energy density

1. By the **local flatness theorem**, at any point one can choose coordinates where $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\lambda}{}_{\mu\nu} = 0$.

   Physically: the frame of a freely falling observer. This is the equivalence principle.

2. In those coordinates the gravitational field is **locally zero**.

3. A **tensor** that vanishes in one chart vanishes in every chart.

CONCLUSION

There is **no stress-energy tensor** for the gravitational field. The best we can build is a **pseudo-tensor** — an object that transforms as a tensor only under linear transformations.

Yet gravitational waves plainly carry energy: they spin binaries down and they move mirrors. Two constructions, both giving the same answer, resolve the tension.

ROUTE A · NOETHER

Treat the metric as a field, apply the canonical construction to a first-order action, symmetrise. Gives the **Landau–Lifshitz pseudo-tensor**.

ROUTE B · AVERAGING

Expand Einstein's equations to **second order** and average over scales large compared with the wavelength. Gives the same expression, but manifestly **non-local**.

WHICH IS THE RESOLUTION?

Route B. The energy of a wave is not defined pointwise — only over regions larger than λ.

The Landau–Lifshitz pseudo-tensor

1. For a Lagrangian depending on **first** derivatives only, the canonical stress-energy is

$$T^{\mu}_{\ \nu} = \sum_n \partial_\nu \psi_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} - \delta^{\mu}_{\ \nu}\, \mathcal{L}$$

conserved by the Euler–Lagrange equations, but not symmetric.

2. Einstein–Hilbert has second derivatives, so use the **ΓΓ (Schrödinger) action**, which differs from it by a surface term:

$$S = -\frac{1}{16\pi}\int \mathrm{d}^4x\, \mathcal{L}, \qquad \mathcal{L} = -\sqrt{-g}\, g^{\alpha\beta}\left(\Gamma^{\mu}_{\ \alpha\beta}\Gamma^{\nu}_{\ \mu\nu} - \Gamma^{\nu}_{\ \alpha\mu}\Gamma^{\mu}_{\ \beta\nu}\right)$$

3. This gives the **Einstein–Schrödinger complex**

$$t^{\mu}_{\ \nu} = -\frac{1}{16\pi\sqrt{-g}}\left[\partial_\nu g_{\alpha\beta}\frac{\partial \mathcal{L}}{\partial(\partial_\mu g_{\alpha\beta})} - \delta^{\mu}_{\ \nu}\mathcal{L}\right]$$

4. Symmetrise it by adding $\partial_\alpha S^{\alpha\mu}{}_{\nu}$ with $S^{[\alpha\mu]\nu} = 0$. The algebra produces the **Landau–Lifshitz superpotential**

$$U^{\mu\alpha\beta\sigma} = (-g)\left(g^{\mu\beta}g^{\alpha\sigma} - g^{\alpha\beta}g^{\mu\sigma}\right)$$

5. whose index antisymmetries give a genuine conservation law:

$$16\pi(-g)\left(T^{\mu\nu} + \tau^{\mu\nu}_{\mathrm{LL}}\right) = \partial_\alpha\partial_\beta U^{\mu\alpha\nu\beta} \implies \partial_\mu\left[(-g)\left(T^{\mu\nu} + \tau^{\mu\nu}_{\mathrm{LL}}\right)\right] = 0$$

SHORTCUT FOR WAVES

Put $g_{\mu\nu} = \eta_{\mu\nu} + h^{\mathrm{TT}}_{\mu\nu}$ in the action and expand to **quadratic** order:

$$S = -\frac{1}{64\pi}\int \mathrm{d}^4x\ \partial_\alpha h^{\mathrm{TT}}_{\rho\sigma}\, \eta^{\alpha\beta}\, \partial_\beta h^{\rho\sigma}_{\mathrm{TT}}$$

$$\tau^{\mathrm{LL}}_{\alpha\beta} = \frac{1}{32\pi}\, \partial_\alpha h^{\mathrm{TT}}_{\rho\sigma}\, \partial_\beta h^{\rho\sigma}_{\mathrm{TT}}$$

The energy of a wave is inherently non-local

1. Work in the **geometric-optics regime**: the perturbation varies on a scale λ much shorter than the background curvature radius L. Define ⟨...⟩ as an average over scales ≫ λ but ≪ L.

2. Expand the metric to second order and the vacuum equations with it:

$$g_{\alpha\beta} = g^B_{\alpha\beta} + \epsilon\, h_{\alpha\beta} + \epsilon^2 j_{\alpha\beta} + \mathcal{O}(\epsilon^3)$$

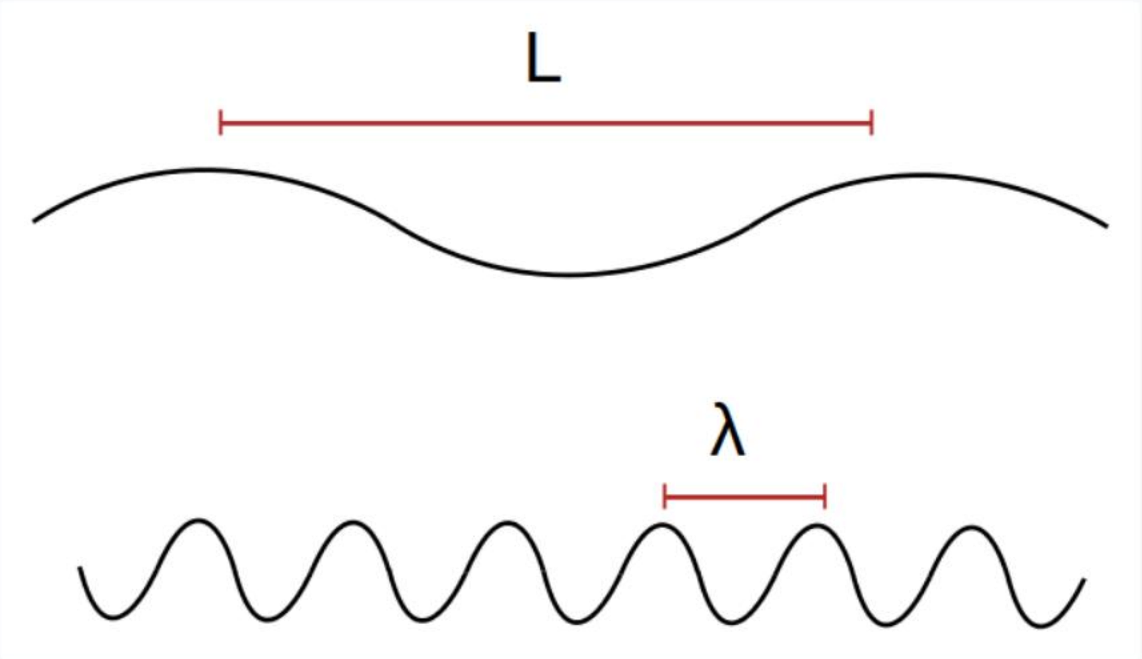
$$0 = G_{\alpha\beta}[g^B] + \epsilon\, G^{(1)}_{\alpha\beta}[h, g^B] + \epsilon^2 \left(G^{(1)}_{\alpha\beta}[j, g^B] + G^{(2)}_{\alpha\beta}[h, g^B] \right)$$

3. Average. $G^{(1)}$ is **linear**, so the average passes through it and kills the oscillatory first-order term. But **$\langle h^2 \rangle \neq 0$**:

$$\left\langle G^{(1)}_{\alpha\beta}[h, g^B] \right\rangle = G^{(1)}_{\alpha\beta}\left[\langle h \rangle, g^B\right] = 0, \qquad \left\langle G^{(2)}_{\alpha\beta}[h, g^B] \right\rangle \neq 0$$

4. Resum the Taylor expansion and read off the backreaction:

$$G_{\mu\nu}\left[g^B + \epsilon^2 \langle j \rangle\right] = 8\pi G\, T^{\mathrm{GW}}_{\mu\nu}, \qquad T^{\mathrm{GW}}_{\mu\nu} = -\frac{1}{8\pi} \left\langle G^{(2)}_{\mu\nu}[h, g^B] \right\rangle$$



λ « L: the wave sees a locally flat background

$$\boxed{T^{\mathrm{GW}}_{\alpha\beta} = \frac{1}{32\pi} \left\langle \nabla_\alpha h^{\mathrm{TT}}_{\rho\sigma}\, \nabla_\beta h^{\rho\sigma}_{\mathrm{TT}} \right\rangle}$$

SAME EXPRESSION, ONE DIFFERENCE

The **average**. Just as the gravitational force vanishes in a falling lift only if the lift is small compared with the Earth, the wave can be gauged away only on scales below λ. Its energy exists, but only over regions **larger than a wavelength**.

The same effect weighs a neutron star

Take a static spherical star, $ds^2 = -B\,dt^2 + A\,dr^2 + r^2 d\Omega^2$, with a perfect fluid. Einstein plus $\nabla_\mu T^{\mu\nu} = 0$ give **TOV**:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho\,\frac{m(r)}{r^2}\left(1+\frac{p}{\rho}\right)\left(1+\frac{4\pi r^3 p}{m(r)}\right)\left(1-\frac{2m(r)}{r}\right)^{-1}$$
$$A(r) = \left(1-\frac{2m(r)}{r}\right)^{-1}, \qquad m(r) = \int_0^r \rho(r')\,4\pi r'^2\,\mathrm{d}r'$$

Outside the star this is Schwarzschild with mass $m(\infty)$ — the **gravitational** mass, the one Kepler's law returns, and the ADM mass of the spacetime.

Now count the matter instead. Baryonic mass and total fluid mass:

$$M_b = m_b \int 4\pi r^2 \sqrt{A}\, n\,\mathrm{d}r, \qquad M_\star = \int 4\pi r^2 \sqrt{A}\,\rho\,\mathrm{d}r$$

Expand the difference in the weak-field limit $m(r)/r \ll 1$:

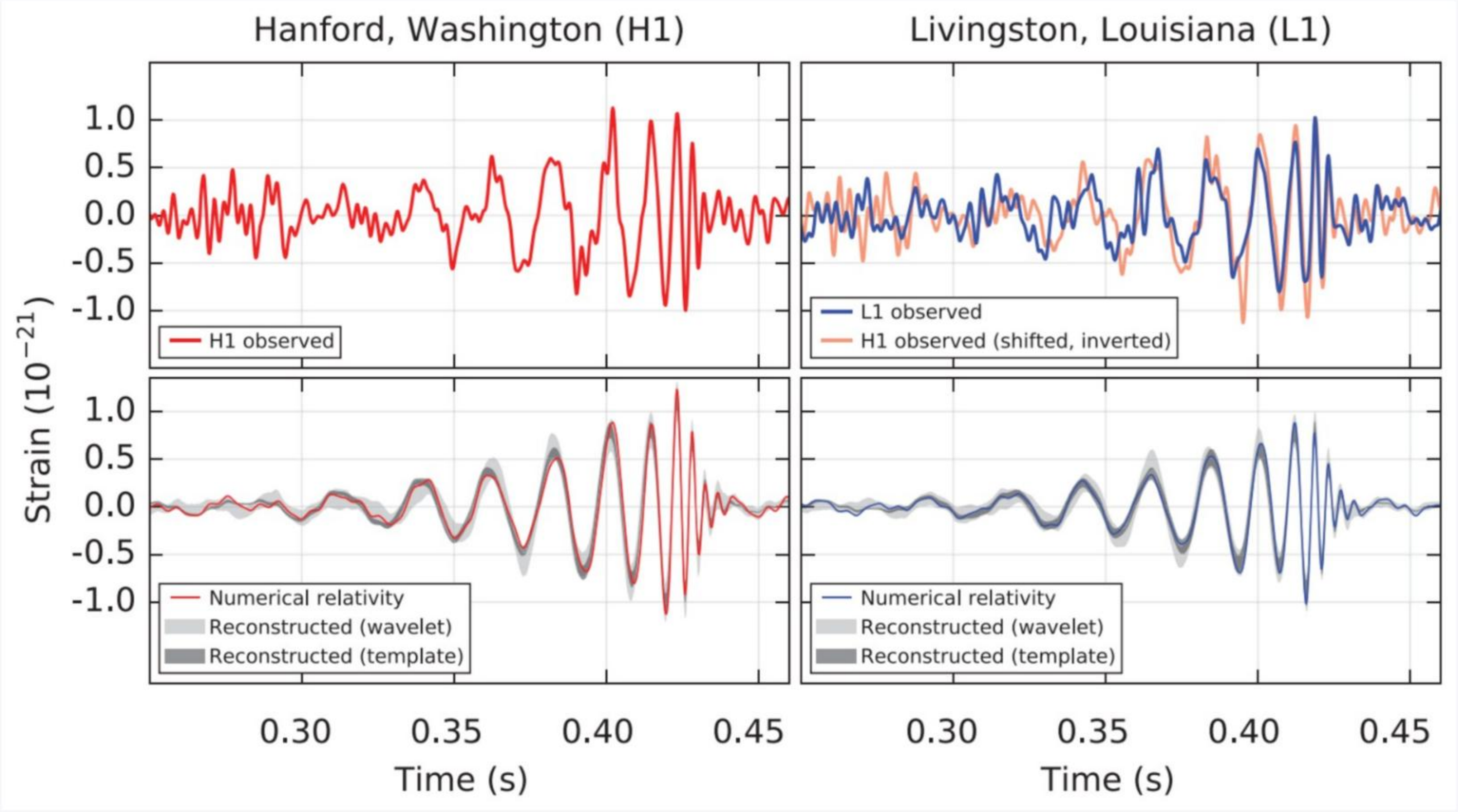
$$m(\infty) - M_\star = \frac{U_{\mathrm{self}}}{c^2}, \qquad U_{\mathrm{self}} = -G\int \mathrm{d}m(r)\,\frac{m(r)}{r}$$

READING

$m(\infty)$ is always **smaller** than $M_\star$, by exactly the Newtonian gravitational **self-energy** — the work needed to disperse the star. For a neutron star this is **10–20%**.

Once again: locally removable, globally real.

After the peak, the signal is a damped sinusoid



GW150914: data (top), reconstruction and numerical relativity (bottom)

The ringdown is visible **in the raw strain**. The claim to justify is that it is a superposition of damped sinusoids whose frequencies depend on **nothing but the mass and spin** of the remnant.

1. **Toy model:** a test scalar on Schwarzschild.
2. **The real thing:** metric perturbations of Schwarzschild — Regge–Wheeler and Zerilli.
3. **The astrophysical case:** Kerr — Teukolsky.

WHERE WE ARE HEADING

A boundary-value problem whose spectrum is **discrete and complex**. Complex means damped — hence *quasi*-normal rather than normal modes.

The toy problem reduces to a Schrödinger equation

1. Free scalar on Schwarzschild:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = 0$$

2. The background is **static and spherically symmetric**, so separate in spherical harmonics and Fourier modes:

$$\varphi = \sum_{\ell m} \frac{R_{\ell m}(r)}{r} Y_{\ell m}(\theta,\phi)\, e^{-i\omega t}$$

The 1/r factor is what removes the first-derivative term.

3. The PDE collapses to a single ODE in the **tortoise coordinate**:

$$r_* \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

$$\boxed{\frac{\mathrm{d}^2 R_{\ell m}}{\mathrm{d}r_*^2} + \left(\omega^2 - V_\ell\right) R_{\ell m} = 0}$$

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right]$$

NOT AUTOMATIC

Reducing the PDE to one ODE is **not** a triviality — it hinges entirely on the ansatz. The Kerr case will need a different basis altogether.

EIKONAL CHECK

For $\ell \gg 1$ this potential reduces to the effective potential for **photon** radial motion, with the impact parameter b = L/E playing the role of ℓ — as it must, since wavefronts follow null geodesics.

The boundary conditions make the spectrum complex

1. Nothing comes **in** from infinity → purely outgoing as $r_* \to +\infty$.
2. Nothing comes **out** of the horizon → purely ingoing as $r_* \to -\infty$.

$$R_{\ell m} \sim e^{+i\omega r_*}\ (r_* \to +\infty), \qquad R_{\ell m} \sim e^{-i\omega r_*}\ (r_* \to -\infty)$$

NOT A NORMAL-MODE PROBLEM

Both conditions **leak**: energy escapes to infinity and down the hole. The problem is not self-adjoint, so the eigenvalues need not be real — and they are not.

$$\omega = \omega_R + i\,\omega_I, \qquad \omega_I < 0 \implies \varphi \propto e^{-i\omega_R t}\, e^{-|\omega_I| t}$$

TWO READINGS OF Ω

The real part is the **ringing frequency**; the imaginary part is the inverse **damping time**. Every mode of Schwarzschild has $\omega_I < 0$ — the geometry is linearly **stable**.

Exercise 4. Approximate the potential by a rectangular barrier, solve in the three regions, impose junction conditions and the ingoing/outgoing conditions. Counting integration constants already shows the spectrum must be **discrete and complex**. Then solve numerically for the first few frequencies.

Build a basis on the two-sphere, then use parity

Classify each block of $h_{\mu\nu}$ by how it transforms on the 2-sphere: h_{tt}, h_{tr}, h_{rr} are **scalars**; h_{tA}, h_{rA} are **vectors**; h_{AB} is a **tensor**.

1. Vector basis from gradients and their duals:

$$Y_A^{\mathcal{E},\ell m} = \partial_A Y_{\ell m}, \qquad Y_A^{\mathcal{B},\ell m} = \epsilon_A{}^B \partial_B Y_{\ell m}$$

2. Tensor basis in the same spirit:

$$\begin{aligned} Y_{AB}^{\mathcal{E},\ell m} &= Y^{\ell m}_{;A;B} - \tfrac{1}{2}\gamma_{AB} Y^{;C}_{\ell m;C} \\ Y_{AB}^{\mathcal{T},\ell m} &= Y_{\ell m}\gamma_{AB} \\ Y_{AB}^{\mathcal{B},\ell m} &= Y^{\mathcal{B},\ell m}_{(A;B)} \end{aligned}$$

$$Y^{\mathcal{E}}, Y^{\mathcal{T}} : (-1)^\ell \quad (\text{even/polar}) \qquad Y^{\mathcal{B}} : (-1)^{\ell+1} \quad (\text{odd/axial})$$

WHY PARITY EARNS ITS KEEP

Even and odd sectors have **different parity**, so they **decouple** when the ansatz is put into the Einstein equations. Two independent problems instead of one coupled system.

3. Expand the gauge generator in the same basis and spend it — the **Regge–Wheeler gauge**:

$$h_2 = \mathcal{H}_0 = \mathcal{H}_1 = G = 0$$

4. Set m = 0 without loss of generality.

Spherical symmetry: a rotation in φ removes it. Note m never appeared in the scalar potential either.

Regge–Wheeler and Zerilli

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r_*^2} + \left(\omega^2 - V\right)\Psi = 0$$

ODD (AXIAL) SECTOR

$$V^- = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right]$$

EVEN (POLAR) SECTOR

$$V^+ = \frac{2}{r^3}\left(1 - \frac{2M}{r}\right)\frac{3\lambda^2 M r^2 + \lambda^2(1+\lambda)r^3 + 9M^2(M+\lambda r)}{(3M+\lambda r)^2}$$

$$\lambda \equiv \frac{(\ell-1)(\ell+2)}{2}$$

ISOSPECTRAL

The two potentials look nothing alike, yet with the same boundary conditions they give **identical** QNM frequencies.

STABLE

All frequencies have **$\omega_I < 0$**: Schwarzschild is linearly stable to gravitational perturbations.

WHERE THE RINGING LIVES

In the eikonal limit both potentials peak at the **unstable circular photon orbit**. ω_R is its orbital frequency; ω_I is the **Lyapunov exponent** of null geodesics there.

Kerr separates — but not in spherical harmonics

THE DIFFICULTY

On Kerr, even the **scalar** equation does not separate in spherical harmonics. That it separates at all is suggested by the **Killing–Yano** tensor, which already separates geodesic motion — and wavefronts follow geodesics in the eikonal limit.

1. Try separation in Boyer–Lindquist coordinates:

$$\varphi = R(r)\,\Theta(\theta)\,e^{im\phi}\,e^{-i\omega t}$$

2. The angular equation is **not** Legendre's:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\left(a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}\right)\Theta=-(Q+m^2)\Theta$$

For a = 0 it reduces to spherical harmonics; for a ≠ 0 it defines the **spheroidal** harmonics. Q is the Carter constant in the eikonal limit.

3. For metric perturbations, work with curvature scalars rather than $h_{\mu\nu}$:

$$\Psi_0=-C_{\mu\nu\lambda\sigma}l^\mu m^\nu l^\lambda m^\sigma,\qquad \Psi_4=-C_{\mu\nu\lambda\sigma}n^\mu m^{*\nu}n^\lambda m^{*\sigma}$$

Ψ_0 and Ψ_4 describe ingoing and outgoing radiation (s = +2 and −2).

4. Expanding in **spin-weighted spheroidal harmonics** gives the master radial equation:

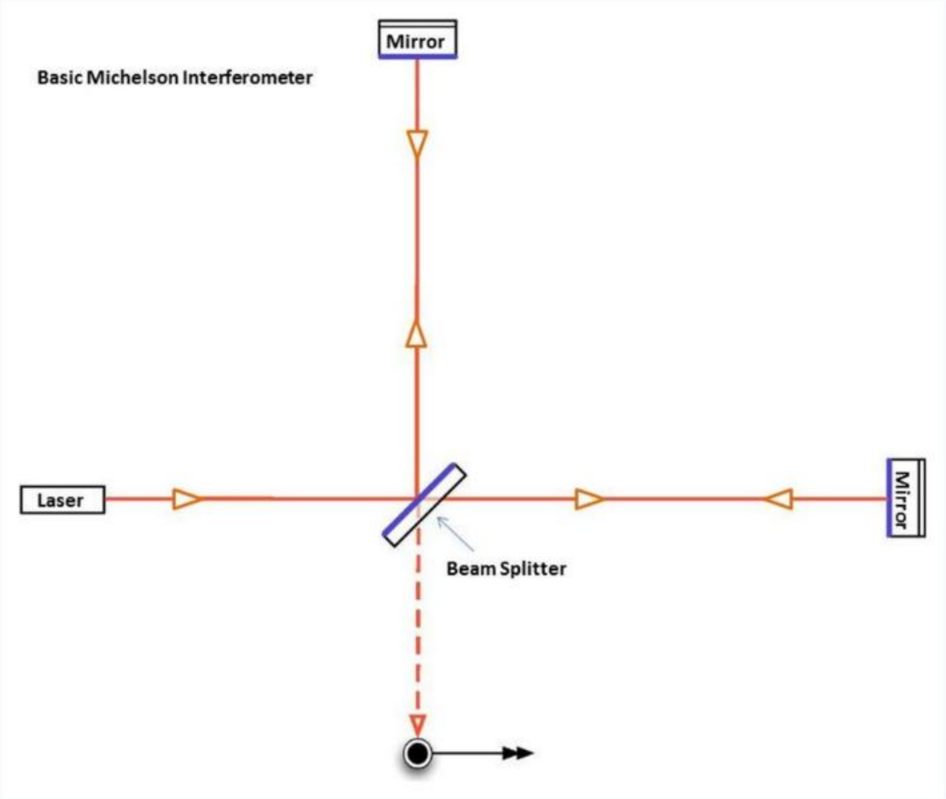
$$\boxed{\Delta\,\partial_r^2 R_{\ell m}+2(s+1)(r-M)\,\partial_r R_{\ell m}+V\,R_{\ell m}=0}$$

$$V=2is\omega r-a^2\omega^2-{}_sA_{\ell m}+\frac{1}{\Delta}\Big[(r^2+a^2)^2\omega^2-4Mam\omega r+a^2m^2+2is\big(am(r-M)-M\omega(r^2-a^2)\big)\Big]$$

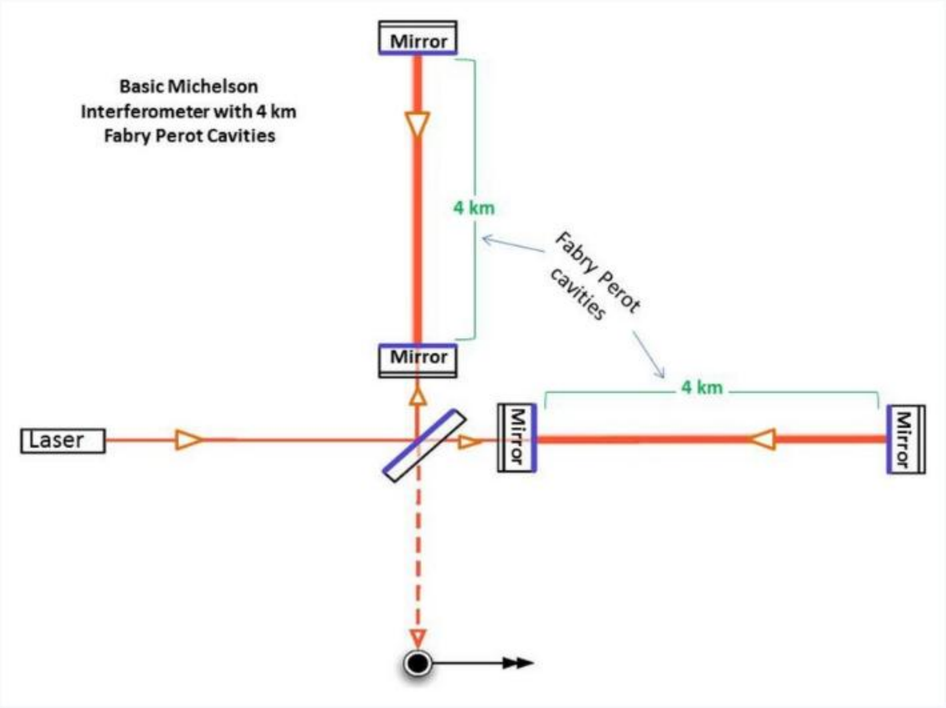
NO-HAIR, AND HOW TO TEST IT

The spectrum depends on **M and a alone**. Measure **two** modes: use the first to fix M and a, then **predict** the second and compare.

What an interferometer actually measures



Michelson — e.g. LISA



Fabry–Pérot — e.g. LIGO

- A wave changes the **two arms differently**, because its angular pattern is not isotropic.
  A single arm would tell you nothing without a reference.
- The time-varying **length difference** becomes a **phase difference** at the photodetector.
- The Fabry–Pérot cavity multiplies the effective armlength — by a factor **~300** for LIGO.

> WHAT WE MUST DERIVE
>
> Two things, in order: the response in the **long-wavelength limit** (§ VIII.A), then what happens when the wavelength is **comparable to the arm** (§ VIII.C).

In TT gauge the mirrors do not move

1. Mirrors in free fall on a spacetime perturbed by a wave:

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{\mathrm{TT}}_{\mu\nu}, \qquad \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

2. Change from proper time to coordinate time. The geodesic equation becomes

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\left(\Gamma^i_{tt} + 2\Gamma^i_{tj}v^j + \Gamma^i_{jk}v^j v^k\right) + v^i\left(\Gamma^t_{tt} + 2\Gamma^t_{tj}v^j + \Gamma^t_{jk}v^j v^k\right)$$

3. For slow mirrors, $v \ll 1$:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} \approx -\Gamma^i_{tt}$$

$$\Gamma^i_{tt} = \tfrac{1}{2}\delta^{ij}\left(2\partial_t h^{\mathrm{TT}}_{jt} - \partial_j h^{\mathrm{TT}}_{tt}\right) = 0$$

because in transverse traceless gauge $h^{\mathrm{TT}}{}_{\mu t} = 0$. A mirror at rest **stays at rest** in these coordinates, for ever, no matter how strong the wave.

RESOLUTION

Coordinates carry no physical meaning in GR. What the laser measures is not coordinate distance but **proper** distance — and that is what changes.

Proper distance, and why the answer is gauge invariant

ROUTE 1 · INTEGRATE THE METRIC

$$L^{\mathrm{proper}} = \int_0^L \mathrm{d}x\,\sqrt{g_{xx}} = \int_0^L \mathrm{d}x\,\sqrt{1+h^{\mathrm{TT}}_{xx}} \simeq L\left[1+\tfrac{1}{2}h^{\mathrm{TT}}_{xx}(t,0)\right]$$

Valid when the perturbation is small *and* its wavelength is much larger than L — that is what lets h come out of the integral.

$$\boxed{\frac{\delta L}{L} \simeq \frac{1}{2}h^{\mathrm{TT}}_{ij}u^i u^j}$$

with u the unit vector along the arm, so the result holds in any frame.

ROUTE 2 · GEODESIC DEVIATION

1. Separation of two neighbouring geodesics:

$$\frac{\mathrm{D}^2 v^\mu}{\mathrm{d}\tau^2} = R^\mu{}_{\alpha\beta\gamma}u^\alpha u^\beta v^\gamma$$

2. In Fermi normal coordinates attached to one mirror,

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(x^2/\lambda^2), \qquad u^\mu = \delta^\mu_t + \mathcal{O}(x^2/\lambda^2)$$

so $\delta x^i = L^i$ *is* the proper separation, and

$$\frac{\mathrm{d}^2 L^i}{\mathrm{d}t^2} = -R_{itjt}L^j$$

3. R_{itjt} is **gauge invariant** at linear order, since the background Riemann tensor vanishes:

$$R_{itjt} = -\tfrac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij} + \partial_i\partial_j\psi + \dot{\Sigma}_{(i,j)} - \tfrac{1}{2}\ddot{\theta}\,\delta_{ij}$$

WHY ONLY ONE TERM SURVIVES

By § II.C the ψ, Σ, θ terms are Newtonian and PN tidal forces, falling as $1/r^3$. Far from the source only h^{TT} remains: $\dfrac{\mathrm{d}^2L^i}{\mathrm{d}t^2} = \dfrac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}L^j + \mathcal{O}\!\left(\dfrac{1}{r^3}\right)$

The detector tensor and the pattern functions

Differencing the two arms in the long-wavelength limit:

$$\frac{\delta L_1 - \delta L_2}{L} = h^{ij}_{\mathrm{TT}} D_{ij} = F_+ h_+ + F_\times h_\times , \qquad D^{ij} = \tfrac{1}{2}\left(u^i u^j - v^i v^j\right)$$

The **detector tensor** D^{ij} is pure geometry — it knows only where the arms point. Contracting it with the wave gives the **pattern functions** F_+, $F_\times$, which encode the antenna's sensitivity across the sky.

$$\Delta\phi = \frac{4\pi\nu}{c}\left(\delta L_1 - \delta L_2\right) = \frac{4\pi\nu L}{c}\left(F_+ h_+ + F_\times h_\times\right)$$

ν is the laser frequency; the factor 2 is the round trip.

Exercise 5. For an interferometer in the (x, y) plane and a wave from (θ, φ), show that

$F_+ = \tfrac{1}{2} \cos 2\psi\,(\cos^2\theta + 1) \cos 2\varphi - \sin 2\psi \cos\theta \sin 2\varphi$

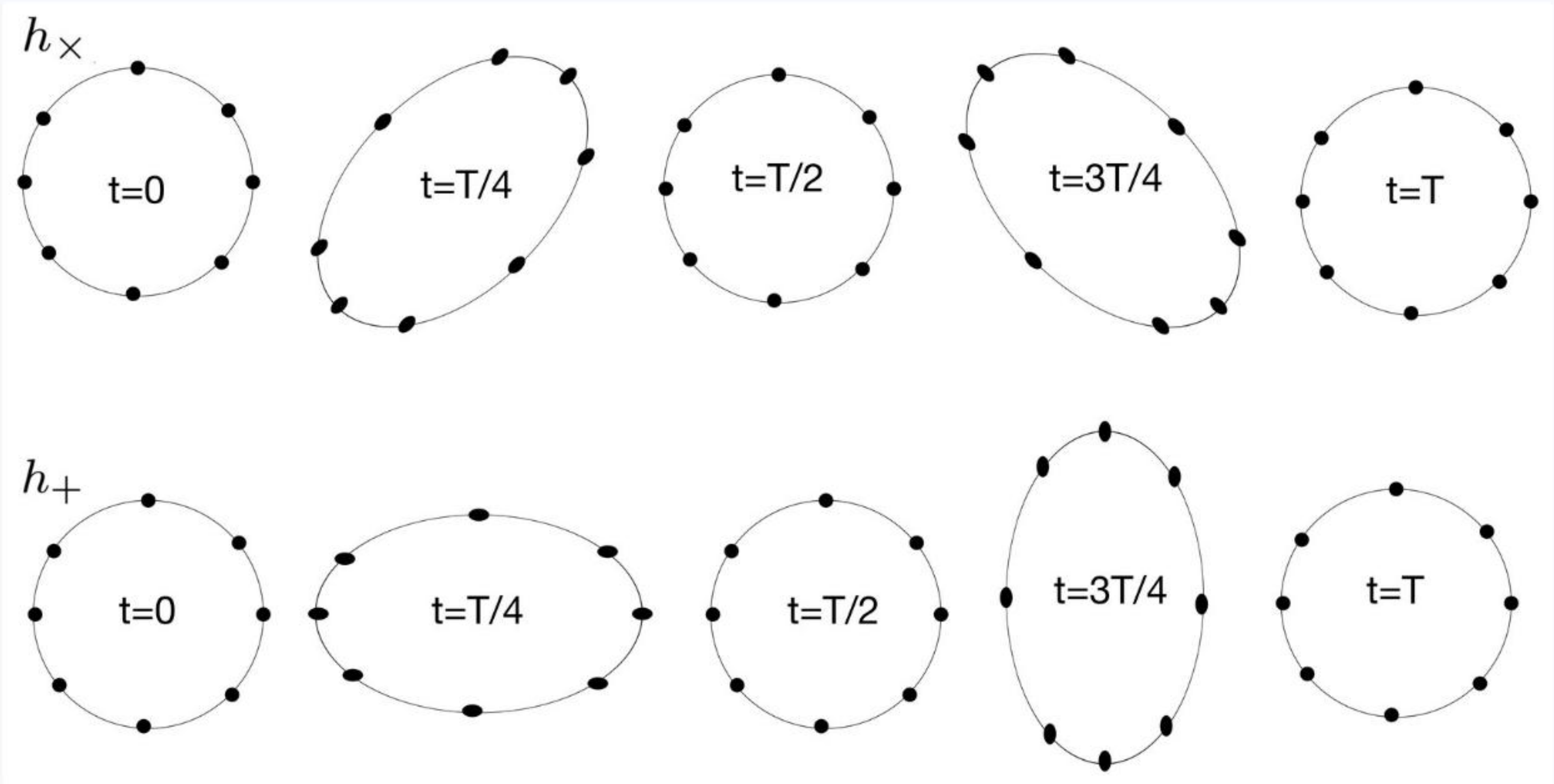
$F_\times = \tfrac{1}{2} \sin 2\psi\,(\cos^2\theta + 1) \cos 2\varphi + \cos 2\psi \cos\theta \sin 2\varphi$

with ψ the **polarisation angle**.

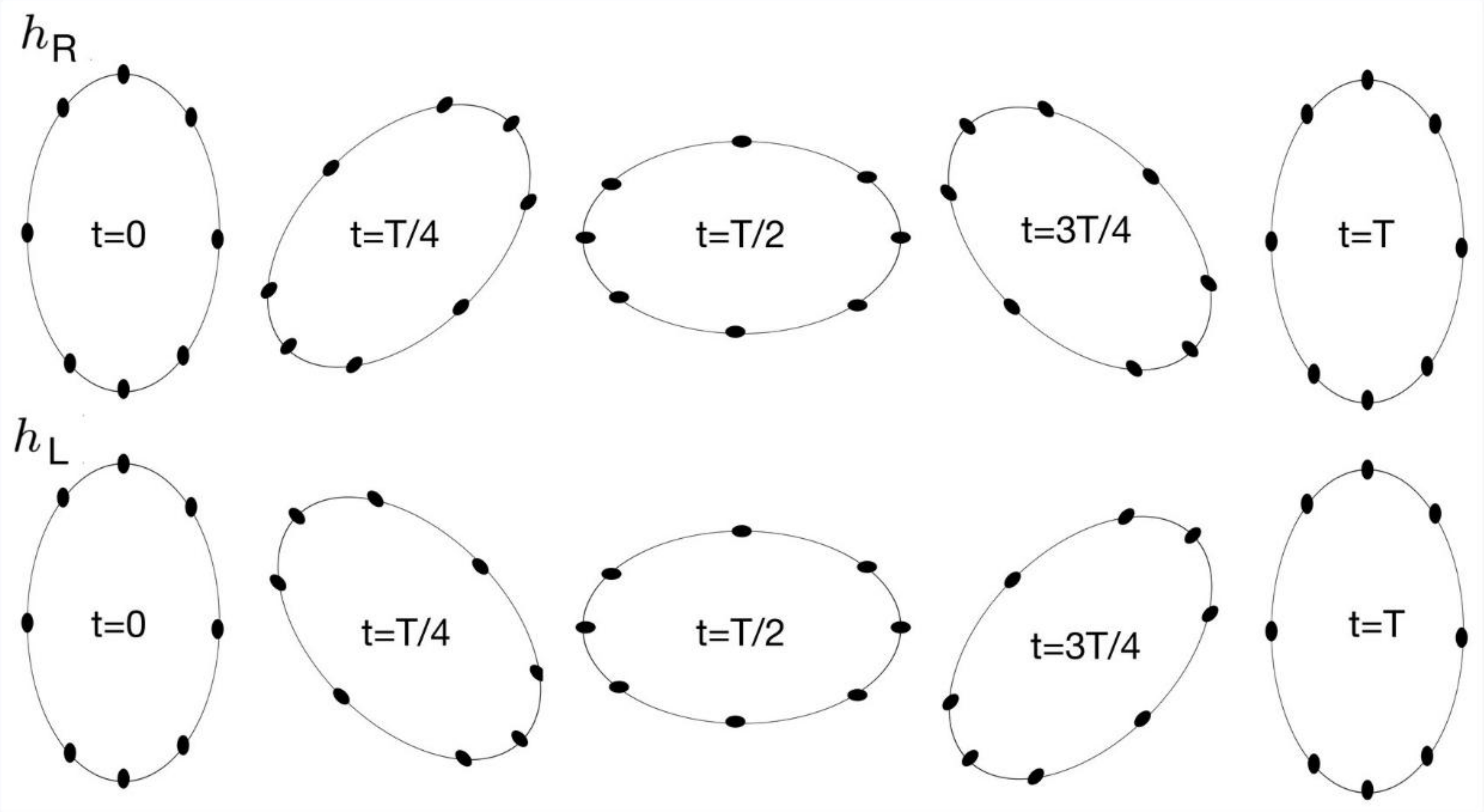
CONSEQUENCE

A single detector cannot separate sky position from polarisation and distance. That degeneracy is broken only by **a network**, which is why detections are quoted from two or more sites.

What the two polarisations do to a ring of particles



Linear polarisation: h_+ and $h_\times$, one period T



Circular polarisation: right and left helicity

Track the photon, not the mirror

1. Take a plane wave along z. The spacetime then has **three Killing vectors**:

$$k_1 = \partial_x, \qquad k_2 = \partial_y, \qquad k_3 = \partial_t + \partial_z$$

2. Projections of the photon wave-vector on them are **conserved** along the null geodesic:

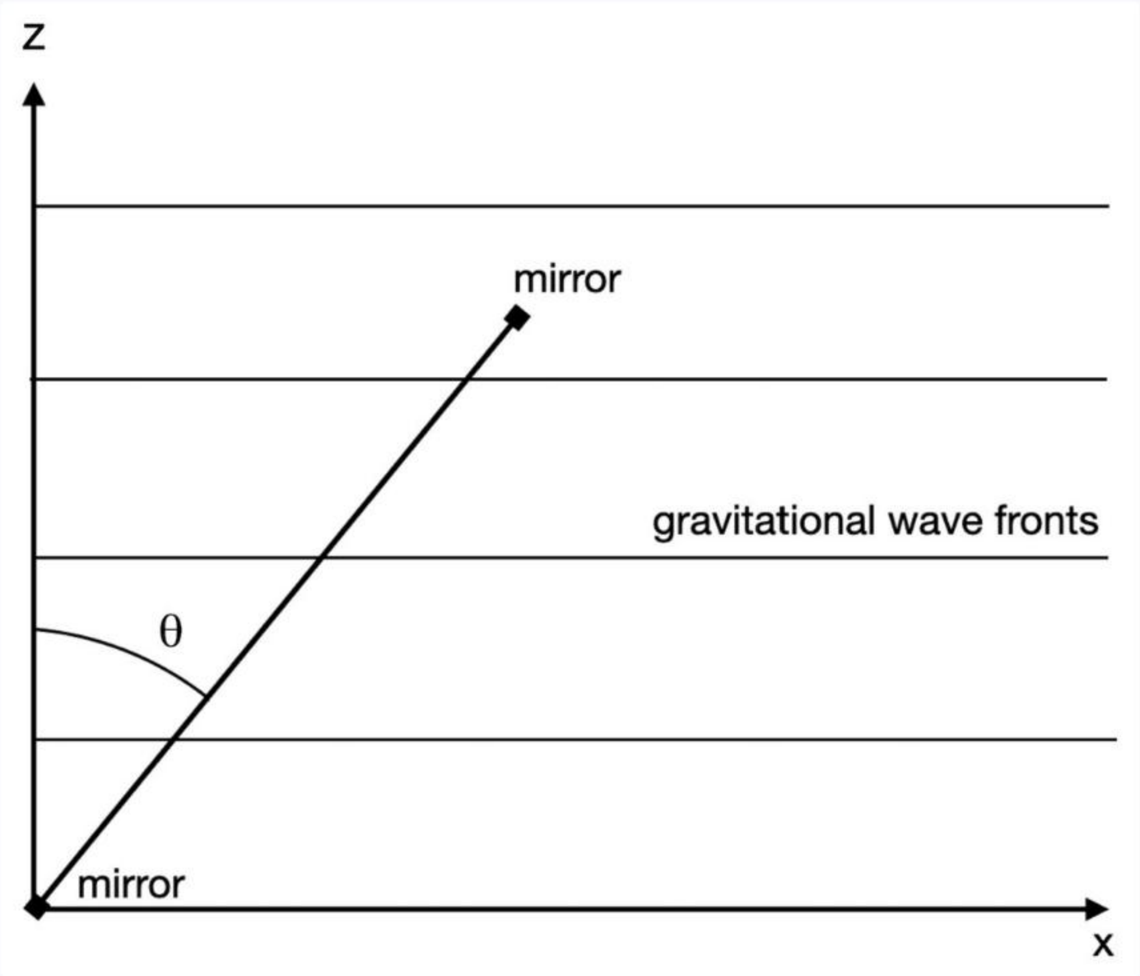
$$-\sigma \cdot k_3 = \nu(1 - \cos\theta) = \text{const}, \qquad \nu \sin\theta\left(1 + \frac{Q}{2}\right) = \text{const}$$

where the combination that matters is

$$Q = \frac{h_{ij}^{\mathrm{TT}} n^i n^j}{1 - \cos^2\theta}$$

3. Follow a photon: mirror 1 → mirror 2 → mirror 1, linearise in h, and collect the **frequency shift over a round trip**:

$$\frac{\Delta\nu}{\nu} = \frac{1}{2}(1+\cos\theta)\,Q(t) - \cos\theta\,Q\left(t + \frac{\tau(1-\cos\theta)}{2}\right) - \frac{1}{2}(1-\cos\theta)\,Q(t+\tau)$$



Round-trip geometry (y = 0)

READ THE THREE TERMS

The wave is sampled at **three different times**: emission, reflection, and return. That is exactly what the long-wavelength limit threw away by evaluating h at a single instant.

The transfer function sets the observing band

1. Take θ = π/2 for clarity:

$$\Delta\nu = \frac{\nu}{2}\left[h(t) - h(t+\tau)\right], \qquad h = h^{\mathrm{TT}}_{ij} n^i n^j$$

2. For a monochromatic wave, integrate to get the phase:

$$\Delta\phi = -2\pi \int \Delta\nu \, \mathrm{d}t = h\left(t + \frac{\tau}{2}\right) \frac{\nu}{f} \sin(\pi f \tau)$$

3. Compare with what an **effective** strain would produce, $\delta L/L = \tfrac{1}{2} h_{\mathrm{eff}}$ and $L = c\tau/2$:

$$\boxed{h_{\mathrm{eff}} = h\, T(f), \qquad T(f) = \frac{\sin(\pi f \tau)}{\pi f \tau}}$$

$$\Delta\phi = \frac{4\pi\nu L}{c}\left[F_+ h_+(f) + F_\times h_\times(f)\right] T(f)$$

LOW FREQUENCY

$f \ll 1/\tau$ gives $T \to 1$ and the response collapses back to the long-wavelength result of § VIII.A. Consistency check passed.

HIGH FREQUENCY

$f \gg 1/\tau$ gives $T \sim 1/(\pi f \tau)$, modulated by zeros where the wave completes whole cycles during a round trip — the detector averages the signal away.

THE DESIGN CONSEQUENCE

The band is set by the armlength, $f_{\mathrm{peak}} \sim c/L$. Kilometre arms put LIGO at hundreds of Hz; LISA's millions of kilometres put it at millihertz. **The instrument's size chooses its sources.**

The five results

$$\boxed{h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 r}\,\mathcal{P}_{ij}{}^{kl}\,\ddot{Q}_{kl}}$$

§ II.D — and, after § III.C, the only linear-theory formula we are entitled to use.

$$\boxed{T_{\alpha\beta}^{\mathrm{GW}} = \frac{1}{32\pi}\langle \nabla_\alpha h_{\rho\sigma}^{\mathrm{TT}}\,\nabla_\beta h^{\rho\sigma}_{\mathrm{TT}}\rangle}$$

§ V — defined only over regions larger than a wavelength.

$$\boxed{\frac{\mathrm{d}^2 R_{\ell m}}{\mathrm{d}r_*^2} + \left(\omega^2 - V_\ell\right) R_{\ell m} = 0}$$

§ VII — with leaking boundary conditions, giving a discrete complex spectrum fixed by M and a alone.

$$\boxed{h_{\mathrm{eff}} = h\,T(f), \qquad T(f) = \frac{\sin(\pi f\tau)}{\pi f \tau}}$$

§ VIII — and with it, why armlength chooses the band.

THE THREAD

Every one of these began by asking what is **physical** rather than what is **convenient**: which parts of the metric are gauge invariant, which conservation law is exact, on what scale energy is defined, which distance the laser really measures.