Relativistic Quantum Information

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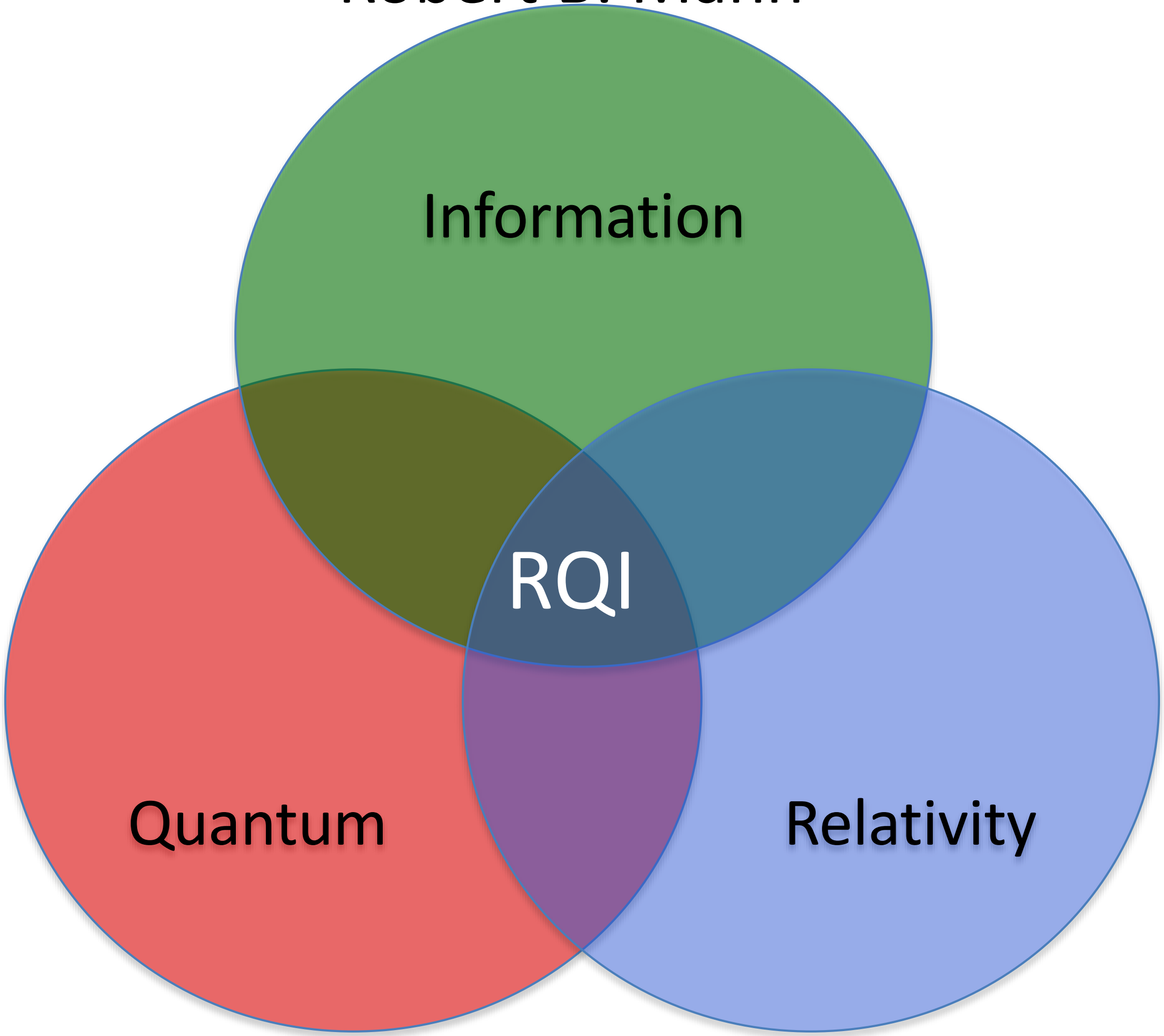


Flat Space in Multiple Guises

Flat Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2 = -dt^2 + dx^2 + \sum_{i=2}^{D} dw^i\, dw^i$$

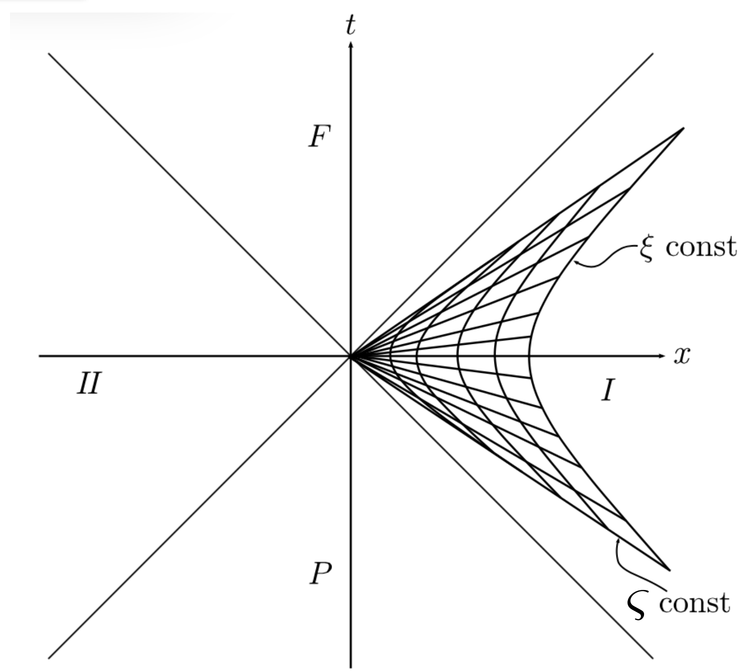
Coordinate Transformations

$$t = \mathrm{X}\sinh(\kappa \mathrm{T}) = \frac{1}{\kappa} e^{\kappa\xi} \sinh(\kappa\varsigma) = \frac{1}{2\kappa}\left(e^{\kappa U} - e^{-\kappa V}\right)$$

$$x = \mathrm{X}\cosh(\kappa \mathrm{T}) = \frac{1}{\kappa} e^{\kappa\xi} \cosh(\kappa\varsigma) = \frac{1}{2\kappa}\left(e^{\kappa U} + e^{-\kappa V}\right)$$

$$\vec{w} = \vec{W}$$

$$\left.\begin{aligned} ds^2 &= -\kappa^2 \mathrm{X}^2 d\mathrm{T}^2 + d\mathrm{X}^2 + \sum_{i=2}^{d} dW^i\, dW^i \\ &= e^{2\kappa\xi}\left(-d\varsigma^2 + d\xi^2\right) + \sum_{i=2}^{d} dW^i\, dW^i \\ &= -e^{\kappa(U-V)} dU\, dV + \sum_{i=2}^{d} dW^i\, dW^i \end{aligned}\right\}$$

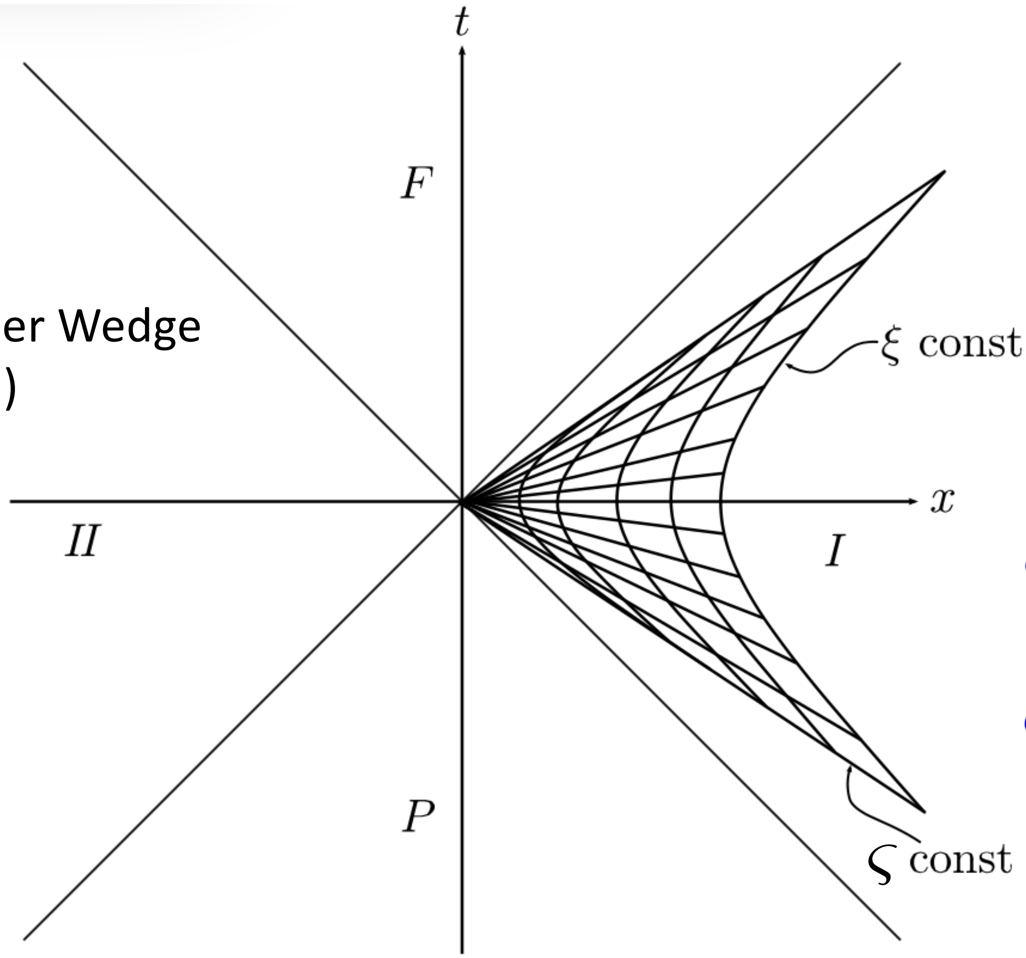


Rindler Space

$$ds^2 = -\kappa^2 \mathrm{X}^2 d\mathrm{T}^2 + d\mathrm{X}^2 + \sum_{i=2}^{d} dW^i\, dW^i$$
$$= e^{2\kappa\xi}(-d\varsigma^2 + d\xi^2) + \sum_{i=2}^{d} dW^i\, dW^i$$
$$= -e^{\kappa(U-V)} dU\, dV + \sum_{i=2}^{d} dW^i\, dW^i$$

Future Rindler Wedge (region F)

Left Rindler Wedge (region II)



Past Rindler Wedge (region P)

Right Rindler Wedge (region I)

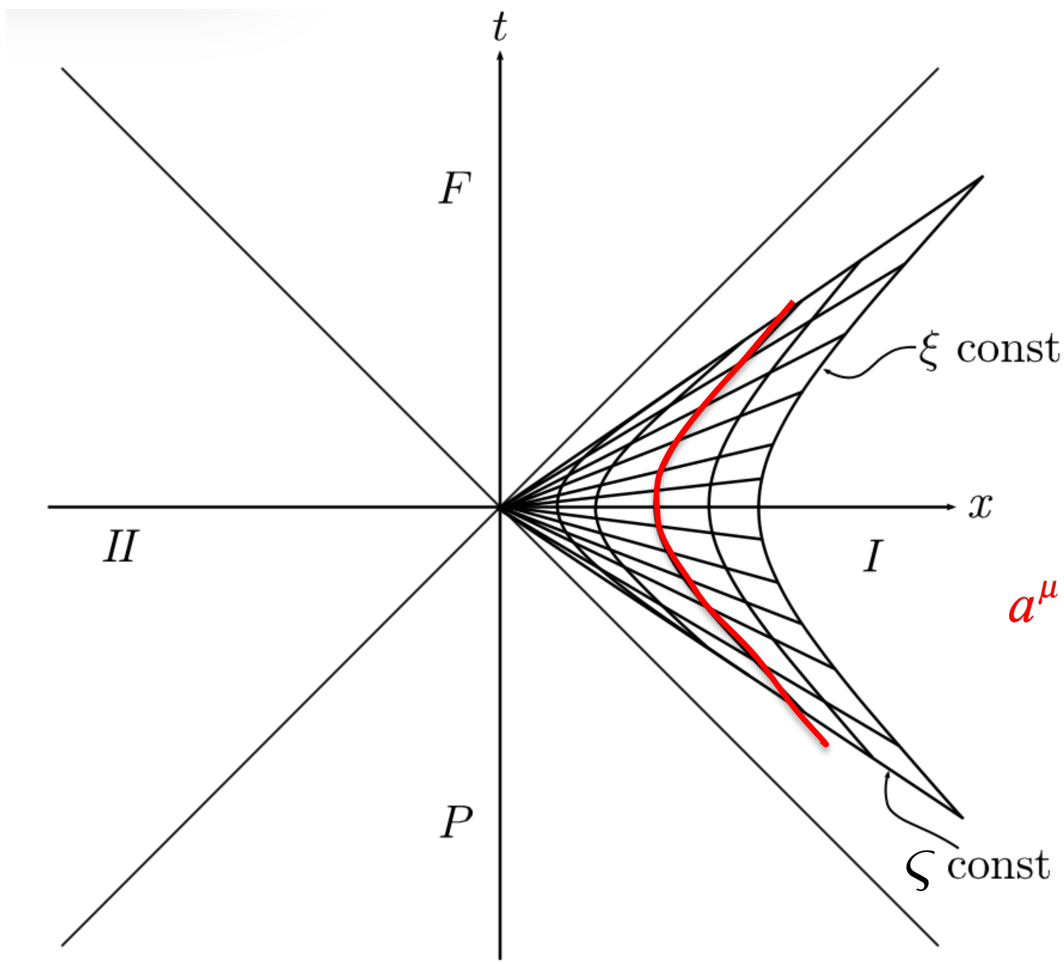
$$\xi = \frac{1}{\kappa}\ln(\kappa \mathrm{X}) = \frac{1}{2\kappa}\ln[\kappa^2(x^2 - t^2)]$$

$$\varsigma = \mathrm{T} = \frac{1}{\kappa}\operatorname{arctanh}(t/x)$$

Rindler Observers

$$ds^2 = e^{2\kappa\xi}(-d\varsigma^2 + d\xi^2) + \sum_{i=2}^{d} dW^i\, dW^i$$

$$z^\mu(\varsigma) = \frac{1}{\kappa} e^{\kappa\xi_0}\left[\sinh(\kappa\varsigma), \cosh(\kappa\varsigma), \vec{w}_0\right]$$



$$v^\mu(\varsigma) = e^{-\kappa\xi_0}\frac{\partial z^\mu}{\partial\varsigma}$$ Proper velocity

$$= \left[\cosh(\kappa\varsigma), \sinh(\kappa\varsigma), \vec{0}\right]$$

$$a^\mu(\varsigma) = e^{-\kappa\xi_0}\frac{\partial v^\mu}{\partial\varsigma}$$ Proper acceleration

$$= \kappa e^{-\kappa\xi_0}\left[\sinh(\kappa\varsigma), \cosh(\kappa\varsigma), \vec{0}\right]$$

$$a_\nu a^\nu = \kappa^2$$

Uniform acceleration for all time!

Acceleration Radiation

Minkowski Spacetime

$$ds^2 = -dt^2 + dx^2 + \sum_{i=2}^{D} dw^i\, dw^i \xrightarrow{t \to i\tau} d\tau^2 + dx^2 + \sum_{i=2}^{D} dw^i\, dw^i$$

Green's function: $(\nabla_x^2 - m^2) G_{\mathcal{M}}(x,y) = \delta^D(x-y) \xrightarrow{t \to i\tau} G_E(\Delta\tau, \vec{x}, \vec{y}) = G_{\mathcal{M}}(i\Delta\tau, \vec{x}, \vec{y})$

$$\text{KMS} \Rightarrow G_E^\beta(\tau) = G_E^\beta(\tau + \beta)$$

Temperature

$$T = 1/\beta$$

Minkowski Observer ⟶ value of β is arbitary

$$d\tau^2 + dx^2 + \sum_{i=2}^{D} dw^i\, dw^i \xrightarrow{(\tau, x, \vec{w}) = \left(\mathrm{X}\sin\left(\frac{2\pi\mathfrak{I}}{\beta}\right), \mathrm{X}\cos\left(\frac{2\pi\mathfrak{I}}{\beta}\right), \vec{W}\right)} \left(\frac{2\pi}{\beta}\right)^2 \mathrm{X}^2 d\mathfrak{I}^2 + d\mathrm{X}^2 + \sum_{i=2}^{d} dW^i\, dW^i$$

Rindler Observer ⟶ value of β is set by the state of motion

$$-\kappa^2 \mathrm{X}^2 d\mathrm{T}^2 + d\mathrm{X}^2 + \sum_{i=2}^{d} dW^i\, dW^i \xrightarrow{\mathrm{T} = i\mathfrak{I}} \kappa^2 \mathrm{X}^2 d\mathfrak{I}^2 + d\mathrm{X}^2 + \sum_{i=2}^{d} dW^i\, dW^i$$

$$\boxed{\beta = \frac{2\pi}{\kappa}}$$

and $G_E(\Delta\tau, \vec{x}, \vec{y}) = G_R(\Delta\mathfrak{I}, \vec{\mathrm{X}}, \vec{\mathrm{Y}}) = G_R(\Delta\mathfrak{I} + \frac{2\pi}{\kappa}, \vec{\mathrm{X}}, \vec{\mathrm{Y}}) = G_R^{2\pi/\kappa}(\Delta\mathfrak{I}, \vec{\mathrm{X}}, \vec{\mathrm{Y}})$

Acceleration Radiation

$$\beta = \frac{2\pi}{\kappa} \Rightarrow T = \frac{\kappa}{2\pi}$$

- A uniformly accelerated detector of scalar particles experiences a thermal bath of scalars (Unruh effect)
- A uniformly accelerated detector/observer in right Rindler wedge is causally disconnected from the left wedge ➔any state (such as the vacuum state) detected by this observer is obtained by tracing over the degrees of freedom in the left wedge
- Yields a mixed state from the perspective of the accelerated observer ➔ entropy coming from entanglement between modes from left and right wedges

The Quantum Vacuum

- Not unique – depends on motion of observer

$$|0\rangle \rightarrow |\Omega\rangle$$

- Has surprising properties that are best explained by introducing 3 notions of operators
  - Field operator: acts over all of spacetime
  - Local operator: zero outside of a bounded region of spacetime
    - Obtain from field operator using smearing functions
  - Quasi-local operator: exponentially decays with distance outside of bounded region of spacetime

The Quantum Vacuum

- Vaccum can be replaced by any state of finite energy

$$|\Xi\rangle = \mathrm{O}|\Omega\rangle \qquad \text{O is some local operator}$$

  – No states of finite energy in a bounded region that can't be obtained from local operators
    - Everything is connected to the vacuum
    - No states are isolated from the vacuum
- No local operator annihilates the vacuum
  – local operators cannot annihilate the vacuum without being completely trivial everywhere
  – if a detector is modeled using local operator(s) then that detector must have a finite probability to `click' in a vacuum → it must have `dark counts'.

The Quantum Vacuum

- Vaccum cannot give zero response

  If $\langle \Omega | \mathrm{W} | \Omega \rangle = 0 \Rightarrow \langle \Psi | \mathrm{W} | \Psi \rangle < 0$ for some $|\Psi\rangle$

  – Standard energy conditions cannot be satisfied

  If $\langle \Omega | T_{\mu\nu} u^{\mu} u^{\nu} | \Omega \rangle = 0 \Rightarrow \langle \Psi | T_{\mu\nu} u^{\mu} u^{\nu} | \Psi \rangle < 0$ for some $|\Psi\rangle$

  – Some states must have locally negative energy density

  – Examples include squeezed states of electromagnetic fields

- Vacuum correlations rapidly decay

$$\left| \langle \Omega | \mathrm{A}_1 \mathrm{A}_2 | \Omega \rangle - \langle \Omega | \mathrm{A}_1 | \Omega \rangle \langle \Omega | \mathrm{A}_2 | \Omega \rangle \right| < \begin{cases} \dfrac{f(\mathrm{A}_1, \mathrm{A}_2, \mathcal{O}_1, \mathcal{O}_2)}{R^2} & \text{massless theory} \\ g(\mathrm{A}_1, \mathrm{A}_2) \exp(-mR) & \text{massive theory} \end{cases}$$

Density Matrices

- Quantum analogue of a probability distribution

Quantum State $|\psi\rangle \Rightarrow |\psi\rangle\langle\psi| \equiv \rho$ ⟸ Density Matrix

Expectation Value: $\mathrm{Tr}[\rho A] = \mathrm{Tr}\big[|\psi\rangle\langle\psi|A\big] = \langle\psi|A|\psi\rangle$

Necessary Properties

Positivity $\rho^\dagger = \rho$

Non-negative eigenvalues

Unit trace $\mathrm{Tr}[\rho] = 1$

Any matrix satisfying these conditions is a valid density matrix

2×2 $$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} = \frac{1}{2}(\mathrm{I} + \vec{r}\cdot\vec{\sigma})$$

Pure State: $\mathrm{Tr}\left[\rho^2\right] = 1$

Mixed State: $\mathrm{Tr}\left[\rho^2\right] < 1$

$\vec{r} = (x, y, z)$

Eigenvalues $\dfrac{(1 \pm |\vec{r}\,|)}{2}$

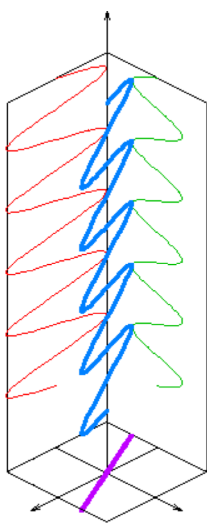
$|\vec{r}\,| = 1 \Rightarrow$ pure

$|\vec{r}\,| < 1 \Rightarrow$ mixed

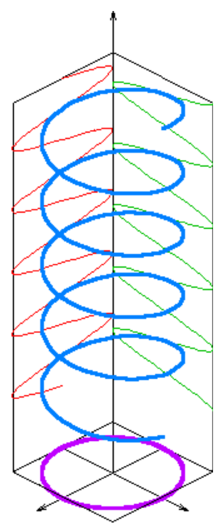
Qubits
The Simplest Quantum Systems

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

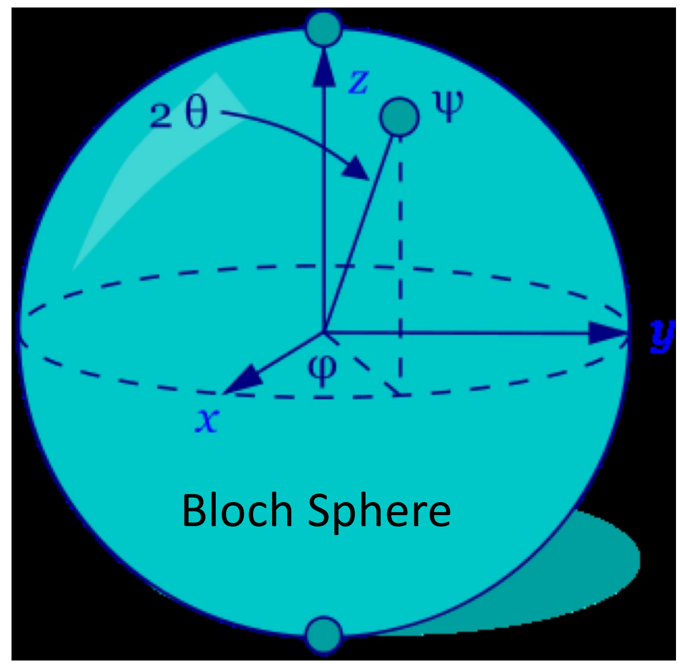
Photon Polarization



Linear



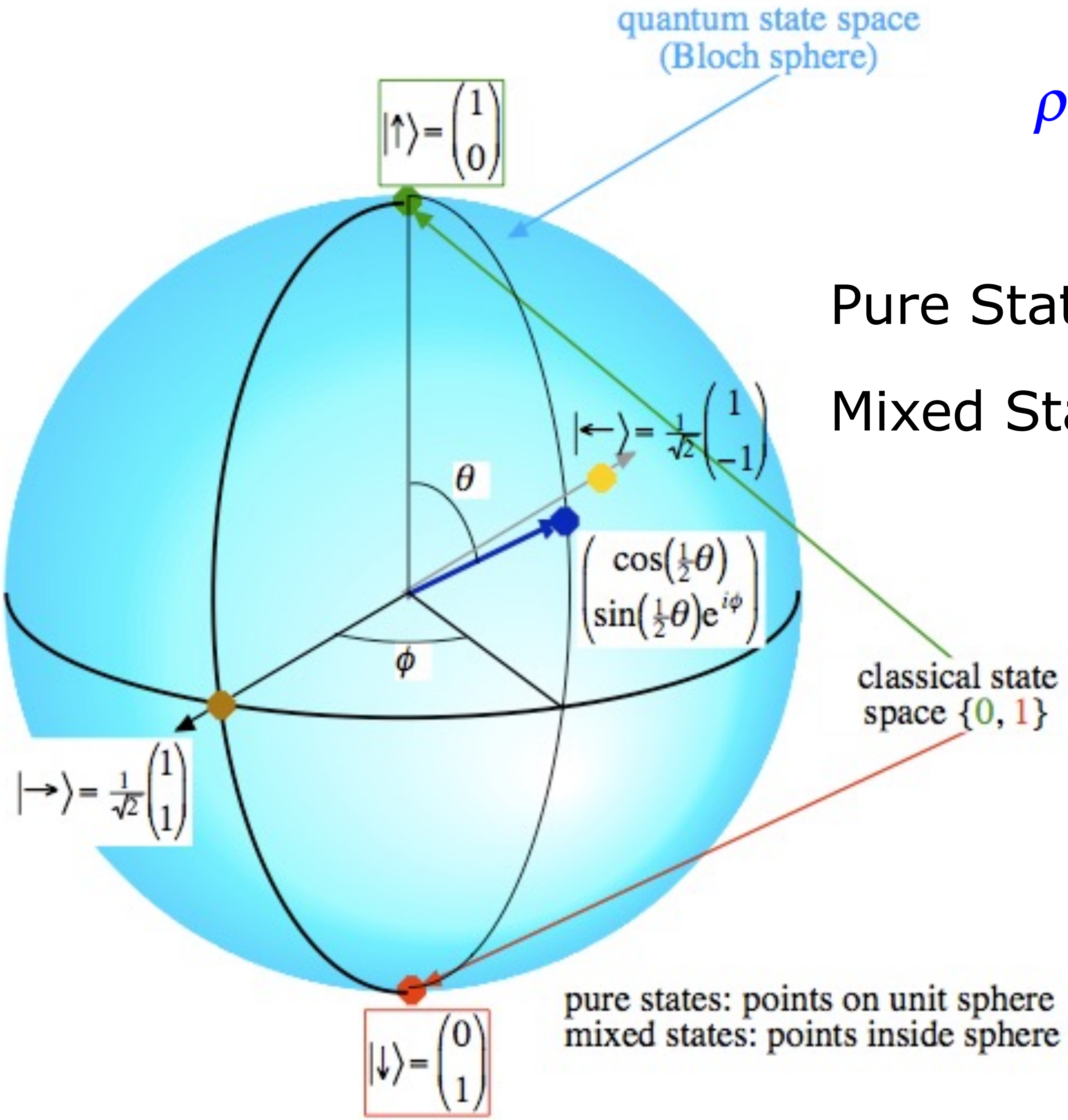
Circular



$= \alpha|$ 

$\rangle + \beta|$ 

$\rangle$



$$\rho = \frac{1}{2}\left(\mathrm{I} + \vec{r}\cdot\vec{\sigma}\right)$$

Pure State: $\mathrm{Tr}\left[\rho^2\right] = 1$

Mixed State: $\mathrm{Tr}\left[\rho^2\right] < 1$

Examples of Pure Density Matrices

Entangled state: $\alpha|0\rangle_A|0\rangle_B+\beta|1\rangle_A|1\rangle_B \Rightarrow \rho=\begin{pmatrix} |\alpha|^2 & 0 & 0 & \alpha\beta^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha^*\beta & 0 & 0 & |\beta|^2 \end{pmatrix}$

$$\mathrm{Tr}\left[\rho^2\right]=1$$

Separable state:

$$\left(\alpha|0\rangle_A+\beta|1\rangle_A\right)\left(\gamma|0\rangle_B+\delta|1\rangle_B\right)$$

$$\Rightarrow \rho=\begin{pmatrix} |\alpha|^2|\gamma|^2 & |\alpha|^2\gamma\delta^* & \alpha\beta^*|\gamma|^2 & \alpha\gamma(\beta\delta)^* \\ |\alpha|^2\gamma^*\delta & |\alpha|^2|\delta|^2 & (\alpha\delta)(\beta\gamma)^* & \alpha\beta^*|\delta|^2 \\ \alpha^*\beta|\gamma|^2 & (\alpha\delta)^*(\beta\gamma) & |\beta|^2|\gamma|^2 & |\beta|^2\gamma\delta^* \\ (\alpha\gamma)^*(\beta\delta) & \alpha^*\beta|\delta|^2 & |\beta|^2\gamma^*\delta & |\beta|^2|\delta|^2 \end{pmatrix}$$

Describing Entanglement

Separable state:

$$|\phi\rangle_{AB} = |\phi\rangle_A \otimes |\psi\rangle_B \Rightarrow$$

Prepare using Local Operations and Classical Communication (LOCC)

Entangled state:
Defined as non-separable

Cannot be prepared by LOCC

$$|\psi\rangle = \alpha|0\rangle_A|1\rangle_B + \beta|1\rangle_A|0\rangle_B \Rightarrow$$

$$|\alpha|^2 + |\beta|^2 = 1$$

Non-local correlations when measured in any basis

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\right) \Rightarrow$$

Maximal correlations when measured in any basis

Quantifying Entanglement

For pure states:

$$|\phi\rangle_{AB} = \sum_{ij} \left(\omega_{ij} |i\rangle_1^A |j\rangle_2^B \right) \Rightarrow |\phi\rangle_{AB} = \sum_{n} \left(\omega_n |n\rangle_1^A |n\rangle_2^B \right)$$

Schmidt decomposition

Measure of entanglement: von-Neumann entropy

$$\rho_{AB} = |\phi\rangle\langle\phi| \Rightarrow S(\rho) = -\mathrm{Tr}\left[\rho \log_2 \rho\right] \Rightarrow S(\rho_{AB}) = 0$$

Reduce

$$\rho_A = -\mathrm{Tr}_{\mathrm{B}}\left[\rho_{AB}\right] \Rightarrow S(\rho_A) = S(\rho_B)$$

Problem: mixed states

$$\rho_{AB} = \sum_{ijkl} \left(\omega_{ijkl} |i\rangle |j\rangle\langle k|\langle l| \right) \Rightarrow$$ No analogous Schmidt decomposition

Entropy no longer quantifies entanglement....

Mixed State Entanglement

Separable state: $\rho_{AB} = \sum_i \alpha_{ij} \rho_A^i \otimes \rho_B^j$

Necessary separability condition:

The partial transpose of the density matrix has positive eigenvalues if the state is separable.

To determine separability is easy

To measure entanglement is not

Mutual information

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

Total correlations quantum + classical

Measures of Entanglement

- Formation Entanglement E_F (Bennett et al)

$$E_F = \min\left\{ \sum_i p_i E(\psi_i) \middle| \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \right\}$$

⇨ The least expected entanglement for any ensemble of pure states with ρ as the density matrix

- Relative entropy of entanglement E_R (Vadral & Plenio)

$$E_R = \min_{\rho \in D}\left\{ Tr\rho\left(\log\rho - \log\rho'\right)\right\}$$

⇨ Entanglement relative to the unentangled mixed state

- Distillable Entanglement E_D (Bennett et al)

$$E_D = \max_T\left\{ \lim_{i\to\infty} \frac{1}{n_i} \dim V_i \right\}$$

⇨ Maximum asymptotic yield of singlets that can be produced from the mixed state via LOCC

Negativity

$$\mathcal{N}(\rho) = \frac{\| \rho^{T_A} \|_1 - 1}{2} = \max\left\{0, -\sum_j \rho_j^-\right\}$$

A measure of how much partial transposition fails to be positive

Logarithmic Negativity

$$\mathrm{E}_{\mathcal{N}}(\rho) = \log \| \rho^{T_A} \|_1 = \log\left(1 + 2\mathcal{N}(\rho)\right)$$

Can show

$$E_D \leq E_R \leq E_F \leq \mathrm{E}_{\mathcal{N}}$$

Upper bound on all entanglement!

In practice $\mathrm{E}_{\mathcal{N}}$ is used to provide a measure of entanglement

Detectors

- Provide an operational means of probing the quantum character of spacetime
- Do not require “global methods”
  - Horizons
  - Eternal acceleration
- Can work in situations of reduced symmetry
- Transferable to curved spacetime

Quantum Detectors

S-Y Lin, B.L.Hu PRD73 (2006) 124018
PRD76 (2007) 064008

Vacuum

$$S_I = \lambda_0 \int d\tau \int d^4x Q(\tau)\Phi(x)\delta^4\left(x^\mu - z^\mu(\tau)\right)$$

Cavity

$$\hat{H} = \overbrace{\Omega_d \hat{a}_d^\dagger \hat{a}_d}^{\text{detector}} + \frac{dt}{d\tau}\sum_n \overbrace{\omega_n \hat{a}_n^\dagger \hat{a}_n}^{\text{field}} + \overset{\text{interaction}}{H_I}$$

$$H_I = \lambda(\tau)(\hat{a}_d e^{-i\Omega\tau} + \hat{a}_d^\dagger e^{i\Omega\tau})\sum_n \left(\hat{a}_n u_n[x(\tau),t(\tau)] + \hat{a}_n^\dagger u_n^*[x(\tau),t(\tau)]\right)$$

E.G. Brown, E. Martin-Martinez, N. Menicucci, RBM PRD87 (2013) 084062
D. Bruschi, A. Lee, I Fuentes J. Phys A46 (2013) 165303

Detector Formalism

$$S=-\int d^4x\sqrt{-g}\left[R+\frac{1}{2}\partial_\mu\Phi(x)\partial^\mu\Phi(x)-\xi R\Phi^2(x)\right]$$

Consider multiple detectors

$$+\sum_D\frac{m_0}{2}\int d\tau\left\{\left[L\left(Q_D(\tau),\dot{Q}_D(\tau)\right)\right]+\sum_D\lambda_D\int d^4x Q_D(\tau)\Phi(x)\delta^4\left(x^\mu-z_D^\mu(\tau)\right)\right\}$$

$$U=\mathcal{T}e^{-i\int dt\left[\sum_D\frac{d\tau_D}{dt}H_{ID}(\tau_D)\right]}$$

$$H_{ID}(\tau)=\chi_D(\tau)\left[e^{i\Omega_D\tau}\sigma_D^+ + e^{-i\Omega_D\tau}\sigma_D^-\right]\Phi[z_D(\tau)]$$

switcher

monopole operator

$$\chi_D=\exp\left(-\frac{(\tau-\tau_D)^2}{2\sigma_D^2}\right)$$

$$\sigma_D^+ := |1_D\rangle\langle 0_D| = |e_D\rangle\langle g_D|$$

$$\sigma_D^+ := |0_D\rangle\langle 1_D| = |g_D\rangle\langle e_D|$$

two detectors

$$\rho_{ij} := \mathrm{Tr}_\Phi\left(U|\Psi\rangle_i\langle\Psi|_i U^\dagger\right)=\begin{pmatrix}1-P_A-P_B & 0 & 0 & X\\ 0 & P_B & C & 0\\ 0 & C^* & P_A & 0\\ X^* & 0 & 0 & 0\end{pmatrix}+\mathcal{O}\left(\lambda^4\right)$$

$$\rho = \begin{pmatrix} 1-P_A-P_B & 0 & 0 & X \\ 0 & P_B & C & 0 \\ 0 & C^* & P_A & 0 \\ X^* & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

$$W(x,x') := \langle 0|\phi(x)\phi(x')|0\rangle$$

$$\tau_D = \gamma_D t$$

$$P_D = \lambda^2 \int_{-\infty}^{\infty} d\tau_D \int_{-\infty}^{\infty} d\tau_{D'}\, \chi_D(\tau_D)\chi_{D'}(\tau_{D'}) e^{-i\Omega_D(\tau_D-\tau_{D'})} W\big(x_D(\tau_D), x_D(\tau_{D'})\big) \quad D = A,B$$

Local excitations

$$C = \lambda^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \frac{d\tau_A}{dt}\frac{d\tau_B}{dt'} \chi_A(\tau_A)\chi_B(\tau_B) e^{-i(\Omega_A\tau_A-\Omega_B\tau_B)} W\big(x_A(t), x_B(t')\big)$$

Local correlations

$$X = -\lambda^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \Big[\frac{d\tau_A}{dt'}\frac{d\tau_B}{dt} \chi_A(\tau_A)\chi_B(\tau_B) e^{-i(\Omega_B\tau_B+\Omega_A\tau_A)} W\big(x_A(t'), x_B(t)\big) + \frac{d\tau_A}{dt}\frac{d\tau_B}{dt'} \chi_A(\tau_A)\chi_B(\tau_B) e^{-i(\Omega_A\tau_A+\Omega_B\tau_B)} W\big(x_B(t'), x_A(t)\big)\Big]$$

Non-Local correlations

Negativity

$$\mathcal{N} = \max\left\{0, \sqrt{|X|^2 - \left(\frac{P_A-P_B}{2}\right)^2} - \left(\frac{P_A+P_B}{2}\right)\right\} + \mathcal{O}(\lambda^4)$$

Concurrence

$$\mathcal{C} = \max\{0, |X| - \sqrt{P_A P_B}\} + \mathcal{O}(\lambda^4)$$

Equal if $P_A = P_B$