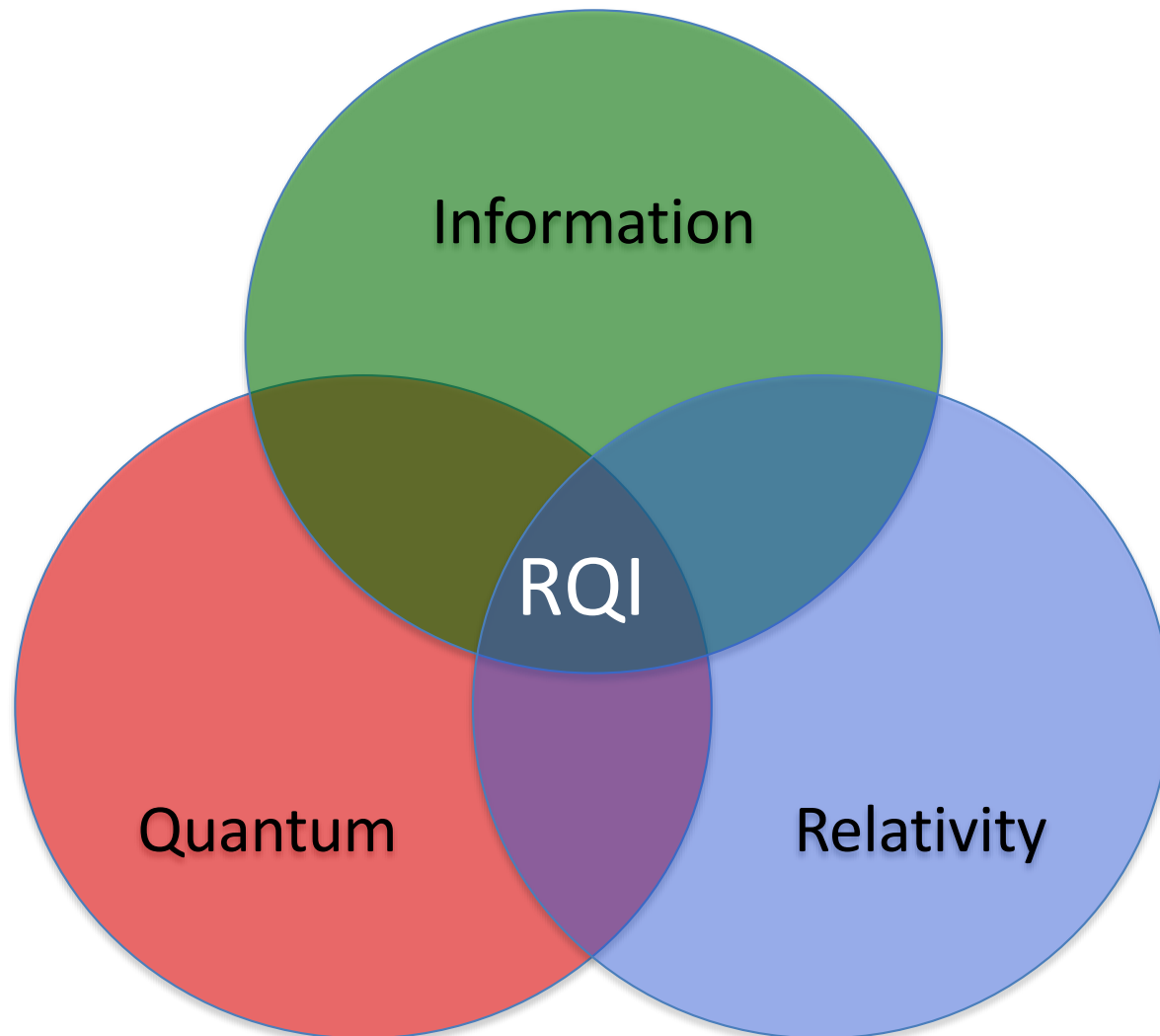


Relativistic Quantum Information

Robert B. Mann



What is RQI?

- How do relativistic effects influence quantum information tasks?
- How do quantum information concepts inform us about the fundamental structure of the universe?



Relativistic Quantum Information

Pre-History

Quantum Field Theory in Curved Spacetime:

- Particles are observer dependent concepts
- Non-inertial detectors experience heat bath
- Black Holes radiate
- de Sitter spacetime is “hot”

Parker and Toms
Davies, Fulling, Unruh
Hawking, Gibbons

Inertial Frames:

- Entanglement is constant between inertially moving observers
- Some degrees of freedom can be transferred to others

Peres, Scudo and Terno
Alsing and Milburn
Gingrich and Adami
Pachos and Solano

Non-Inertial Frames:

- Teleportation fidelity decreases as acceleration increases
- For a bipartite system entanglement degrades as acceleration increases

Alsing/Milburn
PRL **91** 180404 (2003)
Fuentes-Schuller/Mann
PRL **95** 12404 (2005)
Alsing/Fuentes-Schuller/
Mann/Tessier PRA **74**
032326 (2006)

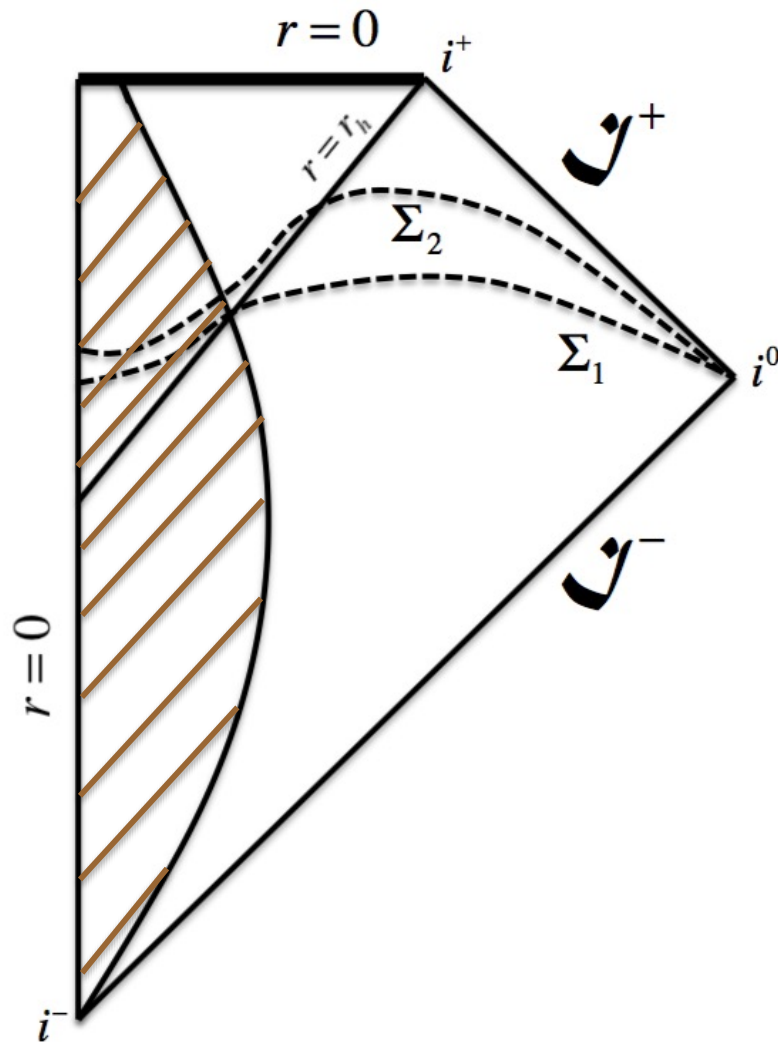
QFT in Curved Spacetime

- Gravity is assumed to be classical
 - Fields are assumed to be quantum
 - Basic idea is to quantize fields in the desired classical spacetime
 - Seems straightforward, but
 - Can't rely on flat-space Poincare symmetry
 - Definition of the vacuum is a subtle concept
 - Different observers disagree on the notion of particle
 - Not clear how to deal with time-dependent spacetimes
 - Relationship between local and global effects
- } **semiclassical approximation**

Key Assumptions

Mathur 2009

- All quantum states are defined on a spacelike slice Σ
intrinsic curvature $R_{abcd}^{(3)} \quad |R_{abcd}^{(3)}| \ll 1 / l_p^2$
extrinsic curvature $K_{\mu\nu} \quad |K_{\mu\nu}| \ll 1 / l_p^2$
- $\exists$ some neighbourhood of Σ where $|R_{\mu\nu\alpha\beta}| \ll 1 / l_p^2$
- $T_{\mu\nu}$ obeys positive energy conditions and $|T_{\mu\nu}| \ll 1 / l_p^2$
- Compton wavelength $\lambda \gg l_p$
- Spacelike slice Σ evolves smoothly:
 $|dN / d\tau| \ll 1 / l_p \quad |dN^a / d\tau| \ll 1 / l_p$



Niceness Conditions

- Quantum properties of spacetime can be neglected
- Gravitational effects of quantum matter can be neglected
- Assumptions remain valid over a finite time interval

Classical Scalar Fields

Action $I = -\frac{1}{2} \int d^D x \sqrt{-g} \left(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + m^2 \phi^2 \right)$

Field Eqn $\nabla^\lambda \nabla_\lambda \phi - m^2 \phi = 0$

Stress Energy $T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{g_{\mu\nu}}{2} \left[\nabla^\lambda \phi \nabla_\lambda \phi + m^2 \phi^2 \right]$

Conjugate Momentum $\pi(x) = \frac{\partial \mathcal{L}_{\text{KG}}}{\partial (u^\mu \partial_\mu \phi(x))} = \sqrt{-g(x)} u^\mu \nabla_\mu \phi(x)$

Hamiltonian $H_{\text{KG}} = \int d^{D-1} x \sqrt{-g(x)} T_{00}$

unit normal to Σ



Scalar Field Quantization

Equal Time Commutation Relation

$$[\varphi(x^0, \vec{x}), \pi(x^0, \vec{y})] = i\hbar \delta(\vec{x}, \vec{y})$$

$$\int d^{D-1}x F(\vec{x}) \delta(\vec{x}, \vec{y}) = F(\vec{y})$$

Covariantly

$$[\varphi(x), \varphi(y)] \equiv i\Delta(x - y)$$

independent of spatial slice
vanishes if (x, y) are spacelike separated

$$= \int \frac{d^D k}{(2\pi)^{D-1}} \delta(k^2 - m^2) \frac{k^0}{|k^0|} \exp(-ik \cdot (x - y))$$

- Local: system is directly influenced only by its immediate surroundings
- Spacelike separated observables made of (φ, π) have no influence on each other
- Measurements made in one region in space and time cannot affect measurements made anywhere outside the light cone of that region

Consequences of Locality

- Spin-statistics theorem: bosons have integer spin, fermions half-integer spin
- Particle/antiparticle symmetry: mass, lifetime, and magnetic moments are same for both a particle and its antiparticle

Positive and Negative Frequency

Conserved Current

$$j^\sigma(\chi, \phi) = i\sqrt{-g}g^{\sigma\nu}(\chi^*\nabla_\nu\phi - (\nabla_\nu\chi^*)\phi) \quad \nabla_\mu j^\mu = 0$$

Inner Product

$$\langle\langle\chi, \phi\rangle\rangle = \int_\Sigma d\Sigma_\nu j^\nu(\chi, \phi) \longrightarrow \begin{aligned} \langle\langle\chi, \chi^*\rangle\rangle &= 0 \\ \langle\langle\chi, \phi\rangle\rangle^* &= -\langle\langle\chi^*, \phi^*\rangle\rangle = \langle\langle\phi, \chi\rangle\rangle \end{aligned}$$

Positive Frequency Solutions

$$\mathcal{Q}_+ := \left\{ \chi, \phi \mid \langle\langle\chi, \chi\rangle\rangle > 0 \text{ and } \langle\langle\chi, \phi^*\rangle\rangle = 0 \right\}$$

Negative Frequency Solutions

$$\mathcal{Q}_- := \left\{ \chi, \phi \mid \langle\langle\chi, \chi\rangle\rangle < 0 \text{ and } \langle\langle\chi, \phi^*\rangle\rangle = 0 \right\}$$

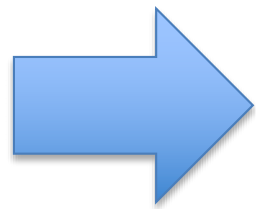
All Solutions

$$\mathcal{Q} = \mathcal{Q}_+ + \mathcal{Q}_-$$

The Vacuum

For a given field φ :

$$\mathbf{a}(\chi) = \langle\langle \chi, \varphi \rangle\rangle \quad \mathbf{a}^\dagger(\chi) = -\mathbf{a}(\chi^*) = -\langle\langle \chi^*, \varphi \rangle\rangle$$



$$[\mathbf{a}(\chi), \mathbf{a}^\dagger(\phi)] = \langle\langle \chi, \phi \rangle\rangle$$

$$[\mathbf{a}(\chi), \mathbf{a}(\phi)] = [\mathbf{a}^\dagger(\chi), \mathbf{a}^\dagger(\phi)] = 0$$

State of lowest energy $\underbrace{|0\rangle}_{\text{The vacuum}}: \quad \mathbf{a}(\chi)|0\rangle = 0 \quad \forall \chi \in \mathcal{Q}_+$

Many particle state $|N\rangle$:

$$\prod_{n=1}^N \mathbf{a}^\dagger(\chi_n)|0\rangle = \mathbf{a}^\dagger(\chi_1)\mathbf{a}^\dagger(\chi_2)\cdots\mathbf{a}^\dagger(\chi_N)|0\rangle$$

Fock space $\mathcal{F}$:
span of all
such states

Mode Decomposition

Decompose the field φ :

$$\varphi(x) = \int d^{D-1}k \left[\phi_k(x) \mathbf{a}_k + \phi_k^*(x) \mathbf{a}_k^\dagger \right]$$

$$\mathbf{a}_k = \langle \langle \phi_k, \varphi \rangle \rangle$$

$$\mathbf{a}_k^\dagger = - \langle \langle \phi_k^*, \varphi \rangle \rangle$$

For a (timelike) Killing vector ξ :

$$i\xi^\mu \partial_\mu \phi_k = \omega_k \phi_k$$

positive frequency modes

$$i \frac{\partial \psi}{\partial t} = E\psi \Rightarrow \psi \sim \exp(-iEt)$$

$$i\xi^\mu \partial_\mu \phi_k^* = -\omega_k \phi_k^*$$

negative frequency modes

$$i \frac{\partial \psi^*}{\partial t} = -E\psi^* \Rightarrow \psi^* \sim \exp(+iEt)$$

$$\xi^\mu \partial_\mu = \frac{\partial}{\partial t}$$

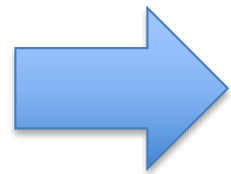
$$\vec{\nabla} \phi_k(x) = \vec{k} \phi_k(x)$$

$$\nabla^\lambda \nabla_\lambda \varphi - m^2 \varphi = 0$$

$$|\vec{k}|^2 + m^2 = \omega_k^2$$

Orthonormal Basis

$$\langle\langle\phi_k, \phi_{k'}\rangle\rangle = -\langle\langle\phi_k^*, \phi_{k'}^*\rangle\rangle = \delta^{(D-1)}(\vec{k} - \vec{k}') \quad \langle\langle\phi_k, \phi_{k'}^*\rangle\rangle = 0$$



$$[\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = \delta^{(D-1)}(\vec{k} - \vec{k}')$$

Minkowski Space

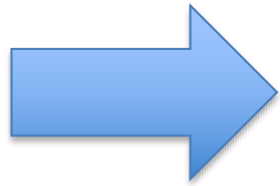
$$\phi_k(x) = \frac{e^{-i\omega_k t + i\vec{k} \cdot \vec{x}}}{\sqrt{2\omega_k (2\pi)^{(D-1)}}}$$

Energy and Momentum

Energy

$$H = \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{(D-1)}} \omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} [\mathbf{a}_k, \mathbf{a}_k^\dagger] \right)$$

eliminate
zero point
contribution



$$H = \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{(D-1)}} \omega_k \mathbf{N}_k \quad \mathbf{N}_k = \mathbf{a}_k^\dagger \mathbf{a}_k$$

Momentum

$$\vec{P} = \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{(D-1)}} \vec{k} \mathbf{a}_k^\dagger \mathbf{a}_k$$

Multiparticle state

$$|n_{k_1}, n_{k_2}, \dots\rangle = \prod_{m=1}^N (\mathbf{a}_{n_k}^\dagger)^{n_k} |0\rangle$$

Energy eigenvalue eqn

$$H |n_{k_1}, n_{k_2}, \dots\rangle = \sum_i n_{k_i} \omega_{k_i} |n_{k_1}, n_{k_2}, \dots\rangle$$

Bogoliubov Transformations

$$Q = Q_+ + Q_- = \tilde{Q}_+ + \tilde{Q}_-$$

$$\begin{array}{llll} Q_+ \Rightarrow |0\rangle & \mathcal{F} = \{|n_k\rangle\} & \xrightarrow{\quad} & |0\rangle \notin \tilde{\mathcal{F}} \\ \tilde{Q}_+ \Rightarrow |\tilde{0}\rangle & \tilde{\mathcal{F}} = \{|\tilde{n}_k\rangle\} & & |\tilde{0}\rangle \notin \mathcal{F} \end{array} \quad \begin{array}{l} \text{space of positive} \\ \text{frequency solutions} \\ \text{is not unique} \end{array}$$

But if $\tilde{\phi} \in \tilde{Q}_+$ then $\tilde{\phi} = \phi + \chi^*$ for some $(\chi, \phi) \in Q_+$.

$$\xrightarrow{\quad} \tilde{\chi}_k(x) = \int d^{(D-1)}k' \left[\alpha_{kk'} \chi_{k'}(x) + \beta_{kk'} \chi_{k'}^*(x) \right] \quad \begin{array}{l} \{\tilde{\chi}_k(x)\} \text{ } \tilde{Q}_+ \text{ basis} \\ \{\chi_k(x)\} \text{ } Q_+ \text{ basis} \end{array}$$

Bogoliubov
Transformation

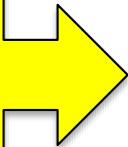
$$\begin{aligned} \alpha_{kk'} &= \langle\langle \tilde{\chi}_k(x), \chi_{k'}(x) \rangle\rangle \\ \beta_{kk'} &= -\langle\langle \tilde{\chi}_k(x), \chi_{k'}^*(x) \rangle\rangle \end{aligned}$$

Two different
particle
interpretations of
field excitations!

Particle Production

$$\tilde{\mathbf{a}}_k = \int d^{(D-1)}k' \left[\alpha_{kk'}^* \mathbf{a}_k - \beta_{kk'} \mathbf{a}_k^\dagger \right] \quad \mathbf{a}_k = \int d^{(D-1)}k' \left[\alpha_{kk'} \tilde{\mathbf{a}}_k + \beta_{kk'}^* \tilde{\mathbf{a}}_k^\dagger \right]$$

Commutation
Relations



$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} \alpha^\dagger & -\beta^T \\ -\beta^\dagger & \alpha^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Number of particles in the $|\tilde{0}\rangle$ vacuum:

$$\langle \tilde{0} | N_k | \tilde{0} \rangle = \langle \tilde{0} | \mathbf{a}_k^\dagger \mathbf{a}_k | \tilde{0} \rangle = \int \frac{d^{(D-1)}k'}{(2\pi)^{(D-1)}} |\beta_{kk'}|^2$$

Nonzero
if $\beta \neq 0$!

Particle Creation and Observer-Dependent Radiation

- Flat space vacuum: state of no particles
 - Minkowski vacuum $\mathbf{a}_k |0\rangle = 0$
- Accelerating Observers: Minkowski vacuum is detected as a thermal bath of particles
- How is this possible since there is no spacetime curvature?
- What is the meaning of a thermal system?

Thermality in Field Theory

Operator O in canonical ensemble

$$\langle O \rangle_\beta \equiv \langle e^{-\beta H} O \rangle = \frac{\text{Tr} [e^{-\beta H} O]}{Z}$$

$$\beta = 1/T$$

Temperature

$$Z = \text{Tr} [e^{-\beta H}]$$

trace over

microstates

of the system

Finite # of
degrees of
freedom

Time Evolution of operators: $\langle A(t) B \rangle = \langle e^{itH} A(0) e^{-itH} B \rangle$

Thermality Condition

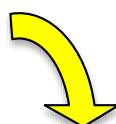
$$\langle A(-i\beta) B \rangle_\beta = \langle BA \rangle_\beta$$

Kubo-Martin-Schwinger
(KMS) Condition

- Thermal state in curved space-time
- Same as above for finite systems
- If KMS holds:
 - free energy is locally minimized
 - any finite system coupled to an infinite system will come into thermal equilibrium with that system

Green's Functions

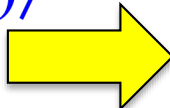
Thermality Condition $\Rightarrow \langle A(-i\beta)B \rangle_\beta = \langle BA \rangle_\beta$

$$\begin{aligned} \text{LHS} &= \langle A(-i\beta)B \rangle_\beta = \langle e^{-\beta H} e^{\beta H} A e^{-\beta H} B \rangle = \langle A e^{-\beta H} B \rangle \\ \text{RHS} &= \langle BA \rangle_\beta = \langle e^{-\beta H} BA \rangle \longrightarrow \langle e^{-\beta H} BA \rangle = \langle A e^{-\beta H} B \rangle \quad \text{KMS} \end{aligned}$$


Time Ordered Green's Function for Scalars

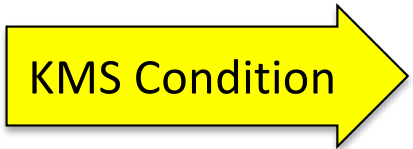
$$\begin{aligned} G_T(x, y) &= \Theta(t_x - t_y) \langle \phi(x) \phi(y) \rangle + \Theta(t_y - t_x) \langle \phi(y) \phi(x) \rangle \\ &\equiv \Theta(t_x - t_y) G_+(x, y) + \Theta(t_y - t_x) G_-(x, y) \end{aligned}$$

complex t -plane



$$\begin{aligned} G_+^\beta(x, y) &= \langle \phi(x) \phi(y) \rangle_\beta \\ G_-^\beta(x, y) &= \langle \phi(y) \phi(x) \rangle_\beta \end{aligned}$$

KMS Condition



$$G_+^\beta(z - i\beta, \vec{x}, \vec{y}) = G_-^\beta(z, \vec{y}, \vec{x})$$

More generally: Any free scalar field whose

Wightman function obeys $\mathcal{G}^\beta(\tau, \vec{x}, \vec{y}) = \mathcal{G}^\beta(\tau + in\beta, \vec{x}, \vec{y})$

on a stationary curved space-time is defined to be in a thermal state

$\mathcal{G}^\beta(z, \vec{x}, \vec{y})$ is holomorphic in complex z plane

except for branch cuts along $(|\Re(z)| > |\vec{x} - \vec{y}|, \Im(z) = n\beta)$

ending in poles at $\Re(z) = |\vec{x} - \vec{y}|$

If space-time is static $\Rightarrow G^\beta(\tau) = G^\beta(\tau + \beta)$

$\rightarrow$ related to thermal states on original (Lorentzian)

spacetime by analytic continuation $t \rightarrow i\tau$

Hot Flat Space

Flat Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2 = -dt^2 + dx^2 + \sum_{i=2}^D dw^i dw^i$$

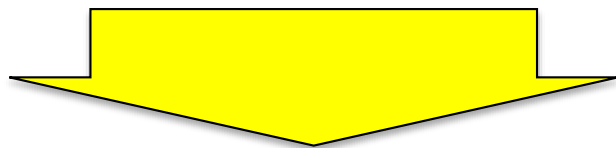
KG equation $\nabla^\lambda \nabla_\lambda \phi - m^2 \phi = 0$ Quantum Vacuum $\mathbf{a}_k |0\rangle = 0$

Green's function: $(\nabla_x^2 - m^2) G_{\mathcal{M}}(x, y) = \delta^D(x - y)$

$$G_{\mathcal{M}}(x, y) = G_{\mathcal{M}}(\Delta t, \vec{x}, \vec{y}) = {}_{\mathcal{M}} \langle 0 | \phi(x) \phi(y) | 0 \rangle \quad \Delta t = x^0 - y^0 = t_x - t_y$$

Euclidean Green's function:

$$G_{\mathcal{M}}(i\Delta\tau, \vec{x}, \vec{y}) = G_E(\Delta\tau, \vec{x}, \vec{y}) = G_E(x, y)$$



$$G^\beta(\tau) = G^\beta(\tau + \beta)$$

← arbitrary

KG Green's function in Minkowski Spacetime can be periodic in imaginary time with any desired period
 → Can have fields at any desired finite temperature

Flat Space in Multiple Guises

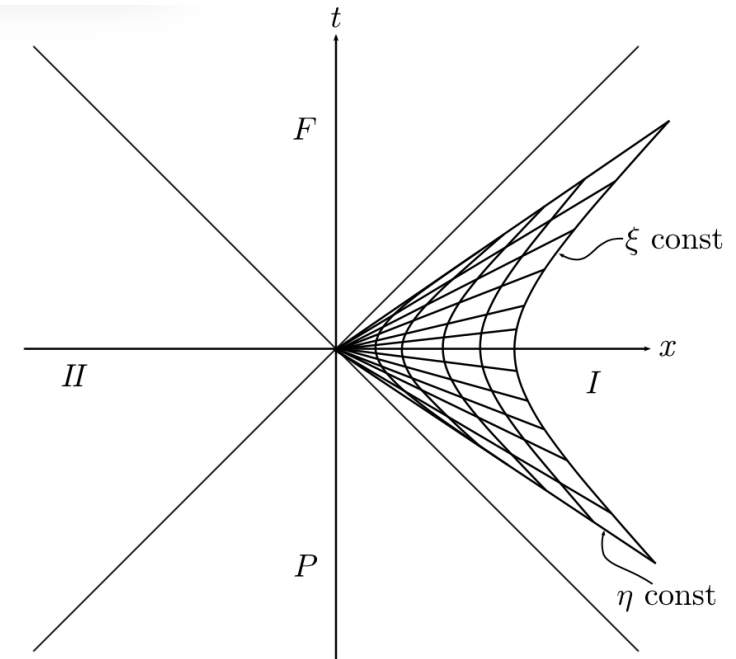
Flat Minkowski Spacetime

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{D-2}^2 = -dt^2 + dx^2 + \sum_{i=2}^D dw^i dw^i$$

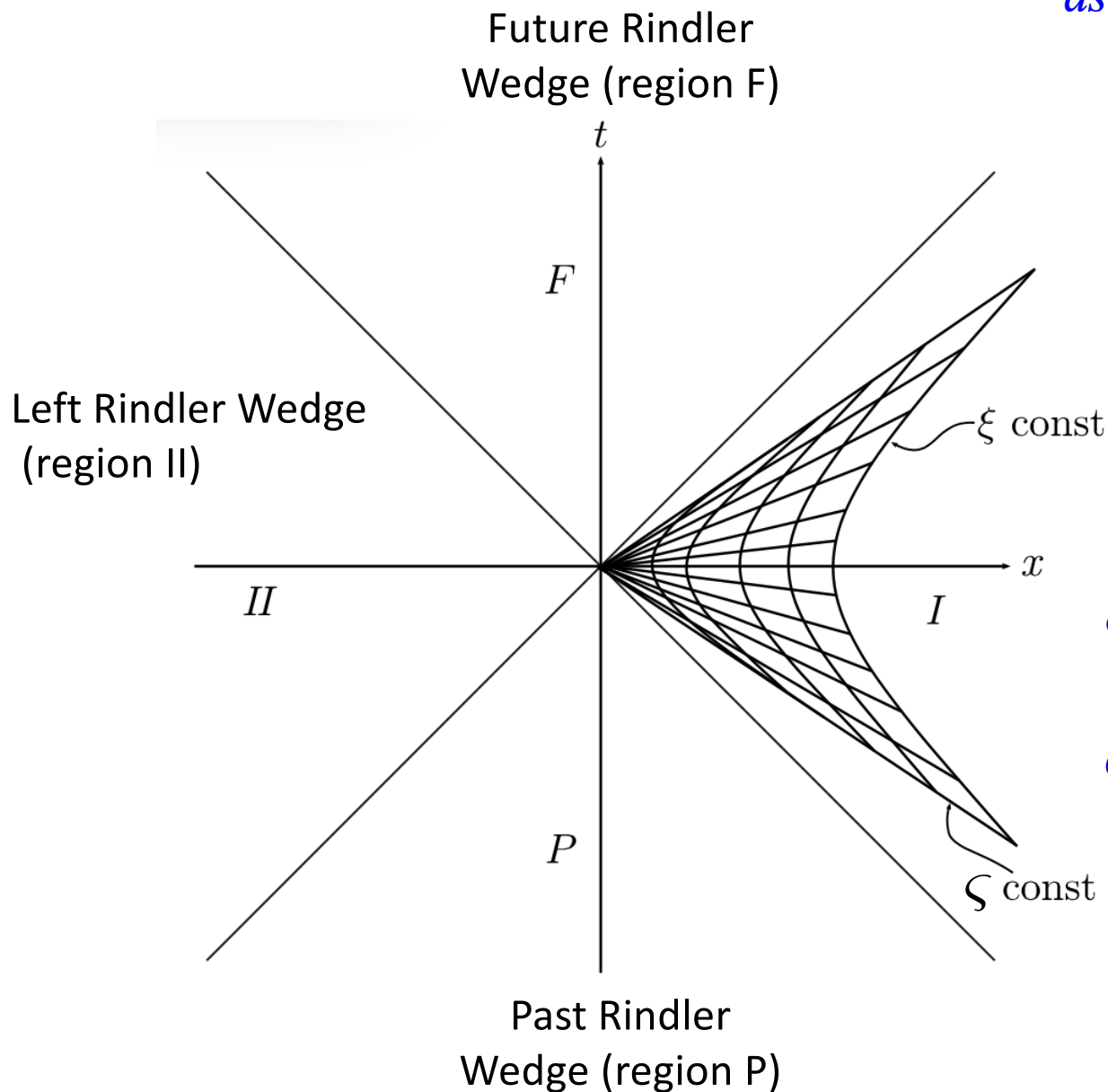
Coordinate Transformations

$$\begin{aligned} t &= X \sinh(\kappa T) = \frac{1}{\kappa} e^{\kappa \xi} \sinh(\kappa \zeta) = \frac{1}{2\kappa} (e^{\kappa U} - e^{-\kappa V}) \\ x &= X \cosh(\kappa T) = \frac{1}{\kappa} e^{\kappa \xi} \cosh(\kappa \zeta) = \frac{1}{2\kappa} (e^{\kappa U} + e^{-\kappa V}) \end{aligned} \quad \vec{w} = \vec{W}$$

$$\begin{aligned} ds^2 &= -\kappa^2 X^2 dT^2 + dX^2 + \sum_{i=2}^d dW^i dW^i \\ &= e^{2\kappa \xi} (-d\zeta^2 + d\xi^2) + \sum_{i=2}^d dW^i dW^i \\ &= -e^{\kappa(U-V)} dU dV + \sum_{i=2}^d dW^i dW^i \end{aligned}$$



Rindler Space



$$\begin{aligned}
 ds^2 &= -\kappa^2 X^2 dT^2 + dX^2 + \sum_{i=2}^d dW^i dW^i \\
 &= e^{2\kappa\xi} (-d\zeta^2 + d\xi^2) + \sum_{i=2}^d dW^i dW^i \\
 &= -e^{\kappa(U-V)} dU dV + \sum_{i=2}^d dW^i dW^i
 \end{aligned}$$

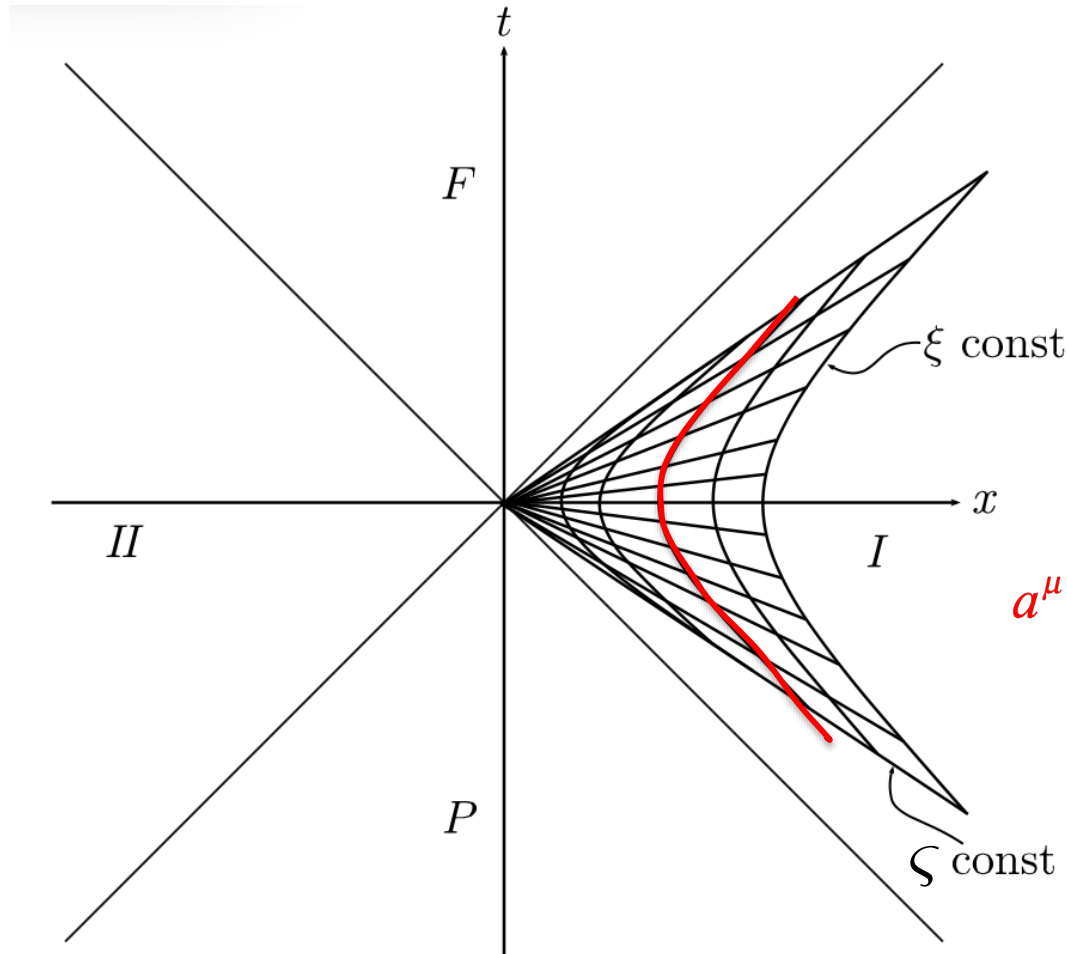
Right Rindler Wedge (region I)

$$\xi = \frac{1}{\kappa} \ln(\kappa X) = \frac{1}{2\kappa} \ln[\kappa^2(x^2 - t^2)]$$

$$\zeta = T = \frac{1}{\kappa} \operatorname{arctanh}(t/x)$$

Rindler Observers

$$ds^2 = e^{2\kappa\xi}(-d\zeta^2 + d\xi^2) + \sum_{i=2}^d dW^i dW^i \quad z^\mu(\zeta) = \frac{1}{\kappa} e^{\kappa\xi_0} [\sinh(\kappa\zeta), \cosh(\kappa\zeta), \vec{w}_0]$$



$$v^\mu(\zeta) = e^{-\kappa\xi_0} \frac{\partial z^\mu}{\partial \zeta} \quad \text{Proper velocity}$$

$$= [\cosh(\kappa\zeta), \sinh(\kappa\zeta), \vec{0}]$$

$$a^\mu(\zeta) = e^{-\kappa\xi_0} \frac{\partial v^\mu}{\partial \zeta} \quad \text{Proper acceleration}$$

$$= \kappa e^{-\kappa\xi_0} [\sinh(\kappa\zeta), \cosh(\kappa\zeta), \vec{0}]$$

$$a_\nu a^\nu = \kappa^2$$

Uniform acceleration for
all time!

Acceleration Radiation

Minkowski Spacetime

$$ds^2 = -dt^2 + dx^2 + \sum_{i=2}^D dw^i dw^i \xrightarrow{t \rightarrow i\tau} d\tau^2 + dx^2 + \sum_{i=2}^D dw^i dw^i$$

Green's function: $(\nabla_x^2 - m^2)G_{\mathcal{M}}(x, y) = \delta^D(x - y) \xrightarrow{t \rightarrow i\tau} G_E(\Delta\tau, \vec{x}, \vec{y}) = G_{\mathcal{M}}(i\Delta\tau, \vec{x}, \vec{y})$

$$\text{KMS} \Rightarrow G_E^\beta(\tau) = G_E^\beta(\tau + \beta)$$

Temperature

$$T = \frac{1}{\beta}$$

Minkowski Observer $\Rightarrow$ value of β is arbitrary

$$d\tau^2 + dx^2 + \sum_{i=2}^D dw^i dw^i \xrightarrow{(\tau, x, \vec{w}) = \left(X \sin\left(\frac{2\pi\mathfrak{I}}{\beta}\right), X \cos\left(\frac{2\pi\mathfrak{I}}{\beta}\right), \vec{W} \right)} \left(\frac{2\pi}{\beta}\right)^2 X^2 d\mathfrak{I}^2 + dX^2 + \sum_{i=2}^d dW^i dW^i$$

Rindler Observer $\Rightarrow$ value of β is set by the state of motion

$$-\kappa^2 X^2 d\mathfrak{T}^2 + dX^2 + \sum_{i=2}^d dW^i dW^i \xrightarrow{\mathfrak{T} = i\mathfrak{I}} \kappa^2 X^2 d\mathfrak{I}^2 + dX^2 + \sum_{i=2}^d dW^i dW^i$$

$$\beta = \frac{2\pi}{\kappa}$$

and $G_E(\Delta\tau, \vec{x}, \vec{y}) = G_R(\Delta\mathfrak{I}, \vec{X}, \vec{Y}) = G_R(\Delta\mathfrak{I} + \frac{2\pi}{\kappa}, \vec{X}, \vec{Y}) = G_R^{2\pi/\kappa}(\Delta\mathfrak{I}, \vec{X}, \vec{Y})$

Acceleration Radiation

$$\beta = \frac{2\pi}{\kappa} \Rightarrow T = \frac{\kappa}{2\pi}$$

- A uniformly accelerated detector of scalar particles experiences a thermal bath of scalars (Unruh effect)
- A uniformly accelerated detector/observer in right Rindler wedge is causally disconnected from the left wedge $\rightarrow$ any state (such as the vacuum state) detected by this observer is obtained by tracing over the degrees of freedom in the left wedge
- Yields a mixed state from the perspective of the accelerated observer $\rightarrow$ entropy coming from entanglement between modes from left and right wedges