Two Nonlinear Degrees of Freedom in Born–Infeld New Massive Gravity

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Abstract

Born–Infeld New Massive Gravity (BINMG) is an all-order curvature extension of New Massive Gravity in three dimensions, whose linearized spectrum around its unique maximally symmetric vacuum contains the two polarizations of a massive spin-two field. Whether an additional scalar degree of freedom appears in the full nonlinear theory has remained an open question. We perform a nonlinear canonical analysis of BINMG using its auxiliary-metric formulation. On the regular branch connected to the maximally symmetric vacuum, the Dirac algorithm yields three first-class diffeomorphism constraints and a second-class pair of scalar constraints. The resulting count gives exactly two local degrees of freedom per spatial point. Thus, nonlinear BINMG propagates no additional Boulware–Deser-type scalar ghost mode on this branch.

1. First-order system
12-dimensional dynamical phase space

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2. Diffeomorphism sector
three first-class gauge chains

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3. Auxiliary scalar sector
one second-class pair

The theory

GR in (3+1)D is not renormalizable, and in (2+1)D is locally trivial. Adding $m^{-2}(R_{\mu\nu}R^{\mu\nu}-3R^2/8)$ gives NMG, which has two massive spin-two degrees of freedom. However, NMG has two maximally symmetric vacua, and generic higher-curvature terms spoil its spectrum. BINMG is the all-order curvature extension of NMG with a unique vacuum:

$$I_{\mathrm{BINMG}}=-\frac{4m^2}{\kappa^2}\int \mathrm{d}^3x\left[\sqrt{-\det\left(g_{\mu\nu}+\frac{\sigma}{m^2}G_{\mu\nu}\right)}-\beta\sqrt{-g}\right].$$

The auxiliary-metric form introduces an independent metric $q_{\mu\nu}$ and the density

$$P^{\mu\nu}:=\sqrt{-q}\,q^{\mu\nu}.$$

Here $G_{\mu\nu}$ is the Einstein tensor. The constant m sets the mass scale, κ is the gravitational coupling, $\sigma=\pm1$ fixes the sign of the Einstein term, and $\beta=1-\lambda_0/2$ contains the bare cosmological parameter λ_0.

What are the ADM variables?

The physical metric is decomposed as

$$g_{\mu\nu}\quad\longrightarrow\quad(N,\,N^i,\,\gamma_{ij}),$$

where N is the physical lapse, N^i the physical shift, and γ_{ij} the spatial metric.
The auxiliary density is decomposed in the same foliation:

$$P^{\mu\nu}\quad\longrightarrow\quad(\nu,\,v^i,\,S^{ij}),$$

where ν is its normal component, v^i its mixed component, and S^{ij} its spatial block.

Legendre-ready form

With the invertible temporary variable $\mathcal{A}^{ij}:=S^{ij}-S\gamma^{ij}$, where $S:=\gamma_{ij}S^{ij}$, integration by parts gives the Legendre-ready bulk density

$$\boxed{\begin{aligned}\mathcal{L}_{\mathrm{Leg}}=&-cK_{ij}\left[\dot{\mathcal{A}}^{ij}-(\mathcal{L}_{\vec N}\mathcal{A})^{ij}\right]\\&+2cK_{ij}\left[\gamma^{ij}D_k(Nv^k)-D^{(i}(Nv^{j)})\right]\\&+Nc\,\mathcal{Q}_K(S,K)+\frac{c\nu}{2}\left(K^2-K_{ij}K^{ij}\right)\\&+N\left[\frac{1}{2}S-\beta\sqrt{\gamma}-\frac{s}{2}v^i(S^{-1})_{ij}v^j-cD_iD_j\mathcal{A}^{ij}\right]\\&+\nu\left[\frac{s-1}{2}+\frac{c}{2}{}^{(2)}R\right].\end{aligned}}$$

Here $c:=\sigma/(2m^2)$, $\gamma:=\det\gamma_{ij}$, $s:=\det S^{ij}$,

$$K_{ij}:=\frac{\dot\gamma_{ij}-(\mathcal{L}_{\vec N}\gamma)_{ij}}{2N},\qquad K:=\gamma^{ij}K_{ij}.$$

and

$$\begin{aligned}\mathcal{Q}_K(S,K):=&\,KS^{ij}K_{ij}-2S^{ij}K_i{}^kK_{kj}\\&-\frac{S}{2}K^2+\frac{3S}{2}K_{ij}K^{ij}.\end{aligned}$$

No spatial derivative acts on K_{ij}, so the dependence on $(\dot\gamma_{ij},\dot{\mathcal{A}}^{ij})$ is algebraic and the Legendre map can be tested.

1. First-order phase space

The two symmetric tensors γ_{ij} and S^{ij} carry six velocities in total. The paper explicitly inverts their six-velocity Legendre map. It has full rank for

$$c:=\frac{\sigma}{2m^2}\neq0.$$

Thus all six velocities are determined by six momenta: no hidden primary constraint occurs in this dynamical block.
The symplectic term gives the final canonical pairs

$$(\gamma_{ij},\rho^{ij}),\qquad(S^{ij},p_{ij}).$$

Here ρ^{ij} and p_{ij} are the momenta conjugate to γ_{ij} and S^{ij}, respectively. Each symmetric tensor has three components in two spatial dimensions, hence

$$N_{\mathrm{phase}}=2(3+3)=12.$$

Reduced canonical action

Since v^i is algebraic when $s:=\det S^{ij}\neq0$, it can be eliminated. The resulting reduced canonical action is

$$I_{\mathrm{red}}=\int\mathrm{d}t\,\mathrm{d}^2x\left[\rho^{ij}\dot\gamma_{ij}+p_{ij}\dot S^{ij}-N\mathcal{H}_0-N^i\mathcal{H}_i-\nu C_0\right].$$

Therefore N and N^i multiply the normal and tangential gravitational densities, while the independent auxiliary variable ν multiplies the additional scalar constraint C_0.

2. Three diffeomorphism gauge chains

Spacetime diffeomorphisms have three independent descriptors in $2+1$ dimensions:

$$\epsilon^\alpha=(\epsilon^\perp,\epsilon^1,\epsilon^2).$$

If the lapse N and shift N^i are retained as canonical variables, their velocities are absent. Their momenta are therefore primary constraints. Preserving them gives the raw secondary constraints:

$$\pi_N\approx0\;\longrightarrow\;\mathcal{H}_0\approx0,\qquad\pi_i\approx0\;\longrightarrow\;\mathcal{H}_i\approx0.$$

Here $\approx$ means equality on the constraint surface.
For an arbitrary spatial vector field ξ^i, define

$$H_{\mathrm{sp}}[\xi]=\int\mathrm{d}^2x\,\left(\rho^{ij}\mathcal{L}_\xi\gamma_{ij}+p_{ij}\mathcal{L}_\xi S^{ij}\right)=\int\mathrm{d}^2x\,\xi^i\mathcal{H}_i.$$

Its canonical brackets give

$$\{\gamma_{ij},H_{\mathrm{sp}}[\xi]\}=\mathcal{L}_\xi\gamma_{ij},\qquad\{S^{ij},H_{\mathrm{sp}}[\xi]\}=\mathcal{L}_\xi S^{ij}.$$

Thus $\mathcal{H}_i$ generate the two tangential diffeomorphisms.
The normal generator requires a correction. Let $\chi_A\approx0$ denote the auxiliary second-class constraints and define

$$\Omega_{AB}:=\{\chi_A,\chi_B\},\qquad\Omega^{AB}\Omega_{BC}=\delta^A{}_C.$$

The representative compatible with their constraint surface is

$$\mathcal{H}_\perp^\star:=\mathcal{H}_0-\{\mathcal{H}_0,\chi_A\}\Omega^{AB}\chi_B.$$

It satisfies

$$\mathcal{H}_\perp^\star\approx\mathcal{H}_0,\qquad\{\mathcal{H}_\perp^\star,\chi_A\}\approx0.$$

The star therefore changes the representative, not the gauge direction.
Diffeomorphism invariance supplies three first-class gauge chains. Since the ADM transformations contain $\dot\epsilon^\alpha$ but no $\ddot\epsilon^\alpha$, Castellani's construction shows that every chain has two stages:

$$\underbrace{\Phi_\perp^\star}_{\substack{\text{primary}\\ \approx\pi_N}}\longrightarrow\underbrace{\mathcal{H}_\perp^\star}_{\text{normal}},\qquad\underbrace{\Phi_i^\star}_{\substack{\text{primary}\\ \approx\pi_i}}\longrightarrow\underbrace{\mathcal{H}_i}_{\text{tangential}},\quad i=1,2.$$

A star denotes projection away from the auxiliary second-class sector when required.

What this sector removes

The three secondary generators $(\mathcal{H}_\perp^\star,\mathcal{H}_1,\mathcal{H}_2)$ are first class on the reduced 12-dimensional dynamical phase space. They remove

$$2\times3=6$$

phase-space directions.

3. The auxiliary scalar chain

Because ν appears linearly in the reduced action, its equation imposes

$$C_0\approx0.$$

Explicitly,

$$C_0=\frac{1}{2}(1-s)-\frac{c}{2}{}^{(2)}R-\frac{1}{2c}\left(p^2-p_{ij}p^{ij}\right),$$

where $s=\det S^{ij}$ and $p=\gamma^{ij}p_{ij}$. So C_0 is a spatial scalar.
Its self-bracket vanishes strongly:

$$\{C_0(x),C_0(y)\}=0.$$

Time preservation means taking its bracket with the total Hamiltonian:

$$\dot C_0(x)=\{C_0(x),H_T\}\approx N(x)C_1(x).$$

Since the physical lapse N remains arbitrary, consistency gives the new secondary scalar constraint

$$C_1\approx0.$$

Why the scalar pair is second class

Define

$$\Delta(x,y):=\{C_0(x),C_1(y)\}.$$

The arbitrary-lapse identity implies that no derivative of the delta function survives:

$$\Delta(x,y)=\partial(x)\,\delta^{(2)}(x-y).$$

At the regular maximally symmetric vacuum,

$$\boxed{\partial_{\mathrm{vac}}=-\frac{\sqrt{a}}{2c}\neq0},\qquad a=\beta^2>0.$$

Thus Δ is invertible there and remains invertible on the connected regular component $\mathcal{U}_{\mathrm{vac}}$ where $\partial(x)\neq0$. Therefore

$$C_0\;\longrightarrow\;C_1\qquad\text{is a second-class chain.}$$

Preserving C_1 fixes the auxiliary variable ν.

Nonlinear degree-of-freedom count

$$N_{\mathrm{dof}}=\frac{1}{2}\left(\underbrace{12}_{\text{phase space}}-2\times\underbrace{3}_{\text{first class}}-\underbrace{2}_{\text{second class}}\right)=\boxed{2}.$$

The three diffeomorphism generators remove six phase-space directions. The scalar pair (C_0,C_1) removes two more.

Only the two massive spin-two polarizations remain.
No Boulware–Deser scalar propagates on $\mathcal{U}_{\mathrm{vac}}$.

Note

The theorem applies only to the regular vacuum-connected component $\mathcal{U}_{\mathrm{vac}}$, where both metrics are nondegenerate, $s\neq0$, and $\partial(x)\neq0$. Degenerate and rank-changing sectors are excluded.

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