The First Law of Black Hole Thermodynamics for General Gravity and Matter Theories

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Abstract

Black hole thermodynamics provides a powerful framework for probing the fundamental structure of gravity. Since the 1990s, a general method for deriving the first law of black hole thermodynamics in diffeomorphism-invariant theories has been developed within the metric formalism. Differential forms, however, offer a natural and efficient language for formulating gravitational theories, particularly in extensions of General Relativity. In this work, we develop a unified prescription for deriving the first law using differential forms defined on spacetime. The framework applies to a broad class of theories, including theories with torsion and generic matter fields, and provides explicit expressions for the corresponding first law.

Wald's Formalism

In any diffeomorphism-invariant theory, black hole entropy is linked to a Noether charge from diffeomorphism symmetry [1]. For an n-form Lagrangian L, variation reads:

$$\delta L = \delta\phi \wedge \mathcal{E}_\phi + \mathrm{d}\theta(\phi, \delta\phi),$$

with equations of motion $\mathcal{E}_\phi$ and boundary term θ.
The conserved current

$$J = \theta(\phi, \mathcal{L}_\xi \phi) - \mathrm{i}_\xi L, \quad \mathrm{d}J = 0,$$

defines a pre-Noether charge Q via $J = \mathrm{d}Q$.
Introducing the pre-symplectic current [2]

$$\Omega(\delta_1, \delta_2) = \int_\Sigma \delta_1\theta(\delta_2) - \delta_2\theta(\delta_1) - \theta([\delta_1, \delta_2]),$$

and taking $\delta_2 = \mathcal{L}_\xi$, δ_1 a variation of the solutions yields

$$\Omega(\delta, \mathcal{L}_\xi) = \int_\Sigma \mathrm{d}\left[\delta Q(\xi) - \mathrm{i}_\xi \theta(\delta)\right],$$

forming the basis of the first law of black hole mechanics.
We extend this framework to first-order variables (tetrads, spin connection), proving the first law's independence from symmetry choice and connection.

Corrected Pre-Symplectic Form

Using vielbeins (e^μ) and spin connection ($\omega^{\mu\nu}$) requires accounting for gauge charges in the first law. This arises from imposing stationarity, $\mathcal{L}_\xi\phi = 0$, on the original dynamical fields, vielbeins and spin connection.
Gauge symmetry implies, within Wald's formalism, a gauge charge $Q_{\mathrm{GT}}(\lambda)$, which contributes an additional boundary term, extending the construction of [3]:

$$Q_{\mathrm{GT}}(e^{\mu a}\delta e^\nu_a) = -\gamma(\delta), \quad \hat{\theta}(\delta) = \theta(\delta) + \mathrm{d}\gamma(\delta).$$

The new boundary term induces a corrected pre-sympelctic form:

$$\hat{\Omega}(\delta_1, \delta_2) = \int_\Sigma \delta_1\hat{\theta}(\delta_2) - \delta_2\hat{\theta}(\delta_1) - \hat{\theta}([\delta_1, \delta_2])$$

This redefinition eliminates the gauge-charge contribution from the presymplectic structure by construction.

First Law of Black hole Thermodynamics

For vielbeins, spin connection and matter field with a non trivial representation of Lorentz group, stationarety implies a gauge transformation.
The Noether charge, which arises from invariance under diffeomorphisms need a correction that includes this gauge transformations. This correction arises naturally from Wald's formalism using $\hat{\Omega}(\delta, \mathcal{L}_\xi)$.

$$\boxed{\int_\mathcal{B} \delta\hat{Q}(\xi) = \int_{i^0} \delta\hat{Q}(\xi) - \mathrm{i}_\xi\hat{\theta}(\delta),}$$

where $\hat{Q}$ is the corrected Noether charge. Here, $\mathcal{B}$ denotes the bifurcation surface where $\xi = 0$, while at spatial infinity, ξ becomes a timelike Killing vector generating the time translation symmetry of the spacetime.

Charges and Boundary Terms Using Differential Forms

We consider diffeomorphism- and local Lorentz-invariant theories formulated in terms of the vielbein e^μ and an independent spin connection $\omega^{\mu\nu}$. The Lagrangian is constructed from the curvature $R^{\mu\nu}$, torsion T^μ, defined

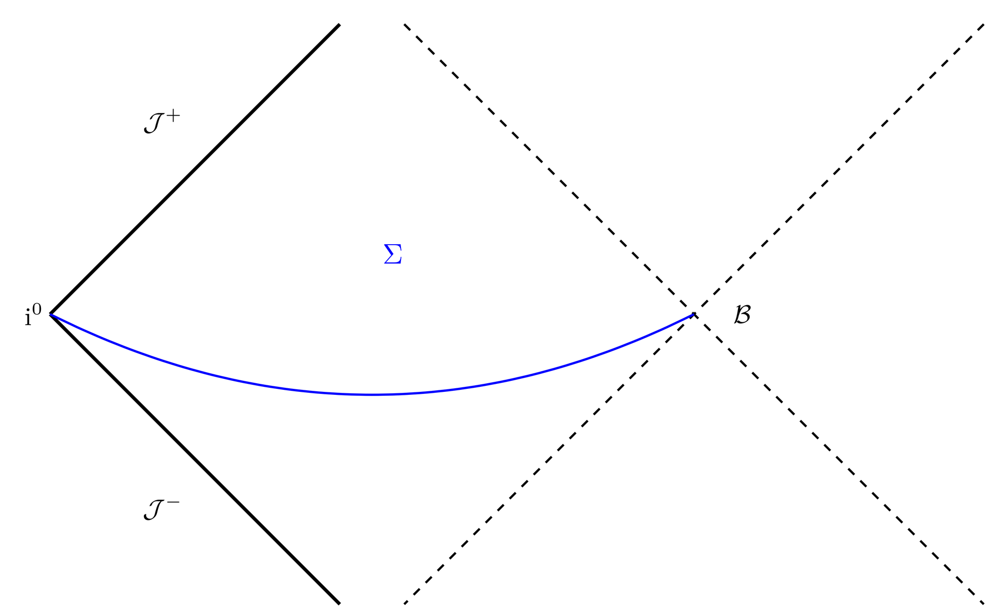
$$R^{\mu\nu} = \mathrm{d}\omega^{\mu\nu} + \omega^\mu{}_\alpha \wedge \omega^{\alpha\nu}, \quad T^\mu = \mathrm{d}e^\mu + \omega^\mu{}_\nu \wedge e^\nu.$$

Matter fields and their covariant derivatives are included in the theory, with Φ^I denoting matter fields that may transform non-trivially under the local Lorentz group. The matter sector may also contain internal gauge fields $A^I{}_J$, with their corresponding covariant derivatives. This framework encompasses a broad class of gravitational theories and their couplings to matter.

For this class of theories, the corrected presymplectic form and Noether charge can be computed explicitly using differential forms, including variations of the dynamical fields, the Hodge operator $\star$, and the covariant exterior derivative D.
Explicit calculations lead us

$$\boxed{\int_\mathcal{B} \delta\left[-\nabla^\mu\xi^\nu\left(H^{(0)}_{\mu\nu} + \mathcal{H}^{(0)}_{\mu\nu}\right)\right] = \int_{i^0} \delta\hat{Q}(\xi) - \mathrm{i}_\xi\hat{\theta}(\delta),}$$

where $H^{(0)}_{\mu\nu}$ and $\mathcal{H}^{(0)}_{\mu\nu}$ are the connection momenta obtained from the variation of the gravitational and matter sectors, respectively. The operator $\mathring{\nabla}$ is the torsion-free covariant derivative in the direcction of the vielbein e^μ. The corrected charge $\hat{Q}(\xi)$ and boundary term $\hat{\theta}(\delta)$ are constructed to be independent of the connection [4].
As a complementary part of this work, we analyze the Hamiltonian structure under suitable fall-off conditions, identifying the Hamiltonian generator directly without explicitly solving for the velocities in terms of the canonical momenta.



Example: Einstein-Cartan-Yang-Mills

The action for this theory is given by

$$S = \frac{1}{2\kappa}\int \epsilon_{\mu\nu\alpha\beta} R^{\mu\nu} \wedge e^\alpha \wedge e^\beta + \frac{1}{2}\int F^I{}_J \wedge \star F^J{}_I,$$

where I, J are indexes associated with an arbitrary $\mathfrak{g}$ Lie algebra. The boundary term and Noether's charge are given by

$$\theta(\delta) = \frac{1}{2\kappa}\delta\omega^{\mu\nu} \wedge \epsilon_{\mu\nu\alpha\beta} e^\alpha \wedge e^\beta + \delta A^I{}_J \wedge \star F^J{}_I,$$
$$Q(\xi) = \frac{1}{2\kappa}\mathrm{i}_\xi\omega^{\mu\nu}\epsilon_{\mu\nu\alpha\beta} e^\alpha \wedge e^\beta + \mathrm{i}_\xi A^I{}_J \star F^J{}_I.$$

Assuming solutions with asymptotic flatness and a bifurcate Killing horizon $\mathcal{B}$, the first law is:

$$\delta\int_\mathcal{B} -\frac{1}{2\kappa}\mathring{\nabla}^\mu\xi^\nu\epsilon_{\mu\nu\alpha\beta}e^\alpha \wedge e^\beta = \int_{i^0}\left[\frac{1}{\kappa}\xi^\mu\epsilon_{\mu\nu\alpha\beta}\delta\omega^{\nu\alpha} \wedge e^\beta + \mathrm{i}_\xi A^I{}_J \star F^J{}_I\right].$$

Conclusions and Future Work

In this work, we have explored the application of Wald's formalism within the first-order approach to gravity, focusing on diffeomorphism-invariant theories. We provided an explicit expression for the associated os the charges that appear in the first law.
The analysis of the presymplectic form $\hat{\Omega}$ shows that it is degenerate along gauge directions in configuration space. Since the presymplectic current provides the geometric structure needed for the construction of quantum field theories [5], this formulation may provide a useful framework for the quantization of gauge theories.

References

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