Teleparallel Regularization of the Gravitational Action

Martin Krššák [1,2] **Michal Stano** [3,4]

[1]CU, Bratislava [2]UST, Hefei [3]IM CAS, Prague
[4]CUNI, Prague

Institute of Mathematics Czech Academy of Sciences

Faculty of Mathematics and Physics Charles University

Motivation

- Finite Euclidean action solutions play an important role in the path integral approach to quantum gravity, black hole thermodynamics, and holography.
- **Complicated construction**: Einstein-Hilbert (EH) action of general relativity (GR) contains second derivatives of the dynamical variable, metric, which requires the addition of the Gibbons-Hawking-York (GHY) boundary term. However, this term generally diverges and thus needs to be regularized $\longrightarrow$ **background subtraction** [2].
- **Alternative approach**: Consider instead the teleparallel equivalent of GR (TEGR), where gravitational action includes only first derivatives of the dynamical variable, tetrad, and thus no extra GHY-like terms are required. However, unlike in GR, the teleparallel spin connection is independent of the tetrad and non-dynamical. We utilize this freedom in choosing the connection to regularize the otherwise divergent TEGR action $\longrightarrow$ **teleparallel regularization**.

Aims

Two main questions: Does the teleparallel regularization agree with the standard methods of GR? Moreover, does it lead to unique results?

- Evaluate the teleparallel gravitational action for the Euclidean Schwarzschild solution
- Compare bulk and quasilocal approaches and explain the role of central singularity
- Address the uniqueness and specify viable asymptotic behavior

Gravitational actions of GR and TEGR

- **GR:** metric and Riemannian connection $\{g_{\mu\nu}, \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu}\}$
- Full gravitational action of GR

$$\mathcal{S}_{\text{GR}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{GHY}} + \mathcal{S}_{\text{counter}} \equiv -\frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\,\overset{\circ}{R} - \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{-\gamma}(\mathcal{K} - \mathcal{K}_0) \tag{1}$$

  featuring the Ricci scalar $\overset{\circ}{R}$, the trace of the extrinsic curvature $\mathcal{K}$, and a counterterm evaluated for the flat reference metric $\mathcal{K}_0$.
- **TEGR:** tetrad and teleparallel connection $\{h^a{}_\mu, \omega^a{}_{b\mu}\}$
- Full TEGR gravitational action

$$\mathcal{S}_{\text{TG}}(h^a{}_\mu, \omega^a{}_{b\mu}) = \frac{1}{2\kappa}\int_{\mathcal{M}} h\,T \tag{2}$$

  constructed from the torsion scalar $T = \frac{1}{4}T^\rho{}_{\mu\nu}T_\rho{}^{\mu\nu} + \frac{1}{2}T^\rho{}_{\mu\nu}T^{\nu\mu}{}_\rho - T^\rho{}_{\mu\rho}T^{\nu\mu}{}_\nu$.

Relating the teleparallel connection to the Riemannian connection allows for the following identification

$$-\overset{\circ}{R} = T + \frac{2}{h}\partial_\mu(h\,T^\mu) \tag{3}$$

Consequently, GR and TEGR yield equivalent field equations, as the total derivative term does not contribute to the dynamics. For vacuum spacetimes, the bulk integral can formally be converted into a quasilocal integral using Stokes' theorem, provided that T^μ is regular throughout the integration domain and all boundary components are included

$$\mathcal{S}_{\text{TG}} \longrightarrow \tilde{\mathcal{S}}_{\text{TG}} = -\frac{1}{\kappa}\int_{\mathcal{M}} \partial_\mu(hT^\mu) \equiv -\frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{\gamma}n_\mu T^\mu \tag{4}$$

Teleparallel regularization

- Teleparallel spin connection is non-dynamical as all the dependence is included in a total derivative term

$$\mathcal{L}_{\text{TG}}(h^a{}_\mu, \omega^a{}_{b\mu}) = \mathcal{L}_{\text{TG}}(h^a{}_\mu, 0) + \frac{1}{\kappa}\partial_\mu(h\,\omega^\mu). \tag{5}$$

- Nevertheless, the boundary term in (5) plays a crucial role when it comes to evaluating the gravitational action.
- We use the **proper frame** prescription [3] defined by the pair $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$ to fix the boundary term (5), which is equivalent to determining the counterterm in GR.
- Matching based on the fact that any arbitrary teleparallel spin connection can be written as Riemannian spin connection of a flat reference tetrad

$$\omega^a{}_{b\mu} = \overset{\circ}{\omega}{}^a{}_{b\mu}({}^0h^a{}_\mu) \tag{6}$$

- Finding the flat reference tetrad ${}^0h^a{}_\mu$ to the full tetrad $h^a{}_\mu$ is thus equivalent to finding the flat reference metric in the method of background subtraction.

Euclidean Schwarzschild black hole in GR

Demonstrate the regularization on the Euclidean Schwarzschild solution

$$ds^2 = f^2 d\tau^2 + \frac{dr^2}{f^2} + r^2 d\Omega^2, \quad f^2 \equiv 1 - \frac{2M}{r}, \tag{7}$$

obtained from the formal analytical continuation of the Lorentzian time $t \longrightarrow -i\tau$.

The value of the gravitational action for the Euclidean Schwarzschild solution ultimately amounts to $\mathcal{S}^E_{\text{GR}} = \frac{\beta M}{2}$.

References

[1] Jose Beltrán Jiménez, Lavinia Heisenberg, and Tomi S. Koivisto.
The canonical frame of purified gravity.
Int. J. Mod. Phys. D, 28(14):1944012, 2019.

[2] G. W. Gibbons and S. W. Hawking.
Action Integrals and Partition Functions in Quantum Gravity.
Phys. Rev., D15:2752–2756, 1977.

[3] Martin Krššák and J. G. Pereira.
Spin Connection and Renormalization of Teleparallel Action.
Eur. Phys. J., C75(11):519, 2015.

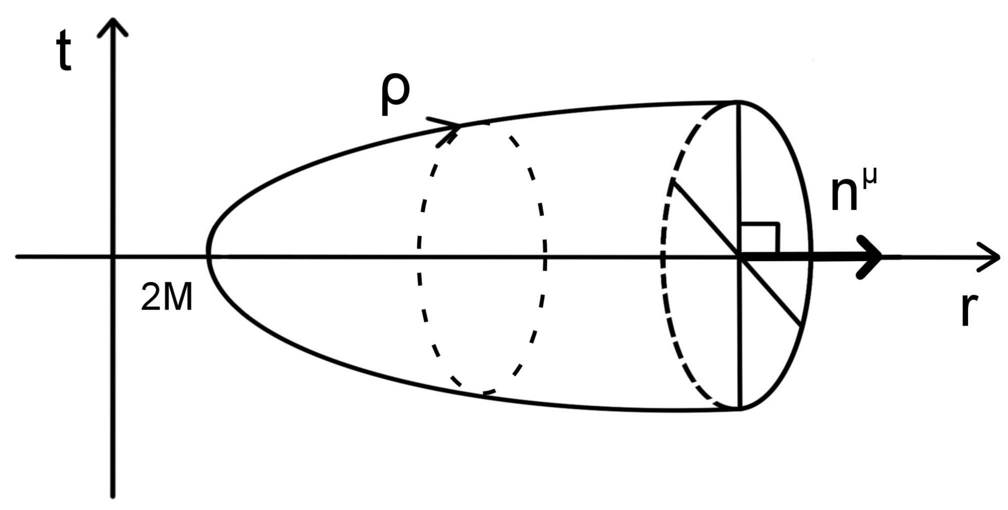

Figure 1. Schematic representation of the Euclidean Schwarzschild manifold $\mathcal{E} \cong \mathbb{R}^2 \times S^2$. Constant-$r$ hypersurfaces have topology $S^1 \times S^2$, with Euclidean time periodically identified as $\tau \sim \tau + \beta$, with $\beta = 8\pi M$.

Bulk versus quasilocal: role of the singularity

Start with the proper diagonal frame

$$h^a{}_\mu = \text{diag}\left(f, f^{-1}, r, r\sin\theta\right) \longrightarrow {}^0h^a{}_\mu = \text{diag}\left(1, 1, r, r\sin\theta\right), \tag{8}$$

where ${}^0h^a{}_\mu$ is obtained from $f|_{r\to\infty} = 1$. The corresponding teleparallel spin connection is then found as

$$\tilde{\omega}^1{}_{2\theta} = -1, \quad \tilde{\omega}^1{}_{3\phi} = -\sin\theta, \quad \tilde{\omega}^2{}_{3\phi} = -\cos\theta. \tag{9}$$

An alternative canonical frame [1] is defined by the vanishing gravitational energy-momentum pseudotensor $t^\mu{}_\nu = 0$. However, in the limit $r \to \infty$, it corresponds to the same flat reference tetrad ${}^0h^a{}_\mu$ in (8), and hence it is matched with the same spin connection (9). Furthermore, since $t^\mu{}_\nu = 0$ implies $t^\mu{}_\mu = T = 0$, the bulk action for the canonical frame vanishes identically, although $T^r \neq 0$.

Frame/prescription	Bulk action	Surface/quasilocal action
Proper diagonal	βM	$\boldsymbol{\beta M/2}$
Canonical	0	$\boldsymbol{\beta M/2}$
GR with GHY subtraction	—	$\boldsymbol{\beta M/2}$

For these two TEGR frames: different bulk values, but the same quasilocal value.

Note the following:

- The bulk action is frame-dependent: βM versus 0.
- For the proper and canonical frames, the quasilocal action agrees with $\mathcal{S}^E_{\text{GR}} = \frac{\beta M}{2}$.
- This agreement does not establish uniqueness: other asymptotically vanishing connection transformations can preserve finiteness while shifting the quasilocal action.

The discrepancy in the two approaches is analogous to a point charge in electrostatics

- Away from the source, $\nabla.E = 0$.
- The bulk integral covers only the regular region.
- The enclosing flux retains the singular contribution.

Asymptotic behavior and uniqueness

For asymptotically flat spacetimes, torsion should not only vanish at $r \to \infty$, but do so **sufficiently rapidly**. Since $h = n_r\sqrt{\gamma} = r^2\sin\theta$, the bulk finiteness requires T to fall-off faster than r^{-3}, i.e. for an integer inverse-power expansion $T = \mathcal{O}(r^{-4})$.

For the quasilocal action, the finiteness requires $T^r = \mathcal{O}(r^{-2})$. A finite, but nonzero quasilocal action demands $T^r = \frac{C}{r^2} + \mathcal{O}(r^{-3}), \quad C \neq 0$.

Uniqueness of the regularizing connection

How unique is the pairing $\{h^a{}_\mu, \tilde{\omega}^a{}_{b\mu}\}$, i.e. is there any redundancy in transformations of the spin connection under fixed tetrad?

Consider r-dependent $SO(4)$ rotation - the Euclidean analogue of a radial boost - with the leading asymptotic term of the angle α parametrised as

$$\alpha \sim \frac{\alpha_0}{r^p}, \quad p > 0 \tag{10}$$

Classified into three categories:

1. $p > 1/2$: leave the action invariant,
2. $p < 1/2$: cause the action to diverge,
3. $p = 1/2$: cause a finite shift of the action due to a change of the leading term in the vectorial torsion

$$T^r(h^a, \tilde{\omega}^a{}_b) = \frac{\alpha_0^2 - M}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \quad \Longrightarrow \quad \Delta\tilde{\mathcal{S}}^E_{\text{TG}} \neq 0 \quad \text{while} \quad \tilde{\mathcal{S}}^E_{\text{TG}} < \infty. \tag{11}$$

Takeaway

$\boxed{\text{Finiteness} \;\not\Longrightarrow\; \text{uniqueness}}$

The critical $\mathbf{p = 1/2}$ class preserves finiteness but shifts the quasilocal action. Thus, although the quasilocal prescription captures the singular contribution, the finiteness of either the bulk or quasilocal action does not by itself make teleparallel regularization unique.