Gradient Catastrophes on Biased Domain Walls

M. Valencia-Villegas

Institute For Theoretical and Mathematical Physics, MSU

Abstract

Domain Walls (DW) must decay. They source Gravitational Wave backgrounds. We show analytically for planar walls:

1. **Thin wall** models — survive (up to cusps).
2. **Thick walls** — can decay via shock waves.

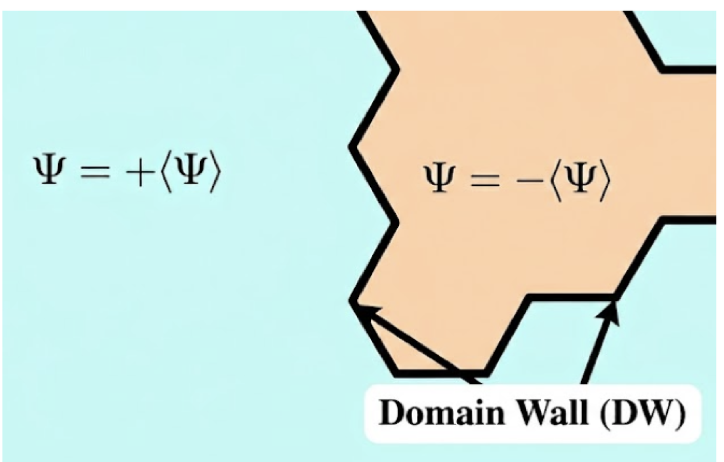
Shock Waves are a new mechanism of energy loss for DW networks and source for Gravitational Wave Backgrounds (SGWB).

Domain Walls (DW)

Domain Walls (DW) develop in models with spontaneous breaking of discrete symmetries *e.g.* Z_2 (with $\epsilon = 0$)

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - \frac{\lambda}{4}(\Psi^2 - \langle\Psi\rangle^2)^2 + \epsilon\frac{\Psi^{2n+1}}{2\langle\Psi\rangle^{2n+1}}\right]$$

- Static 1+1 kink solution: $\Psi(z)|_{\epsilon=0} = \langle\Psi\rangle\ \tanh\left(\sqrt{\frac{\lambda}{2}}\ \langle\Psi\rangle\ z\right)$
- DW are embeddings of kinks in 1+3, separating $\Psi = \pm\langle\Psi\rangle$



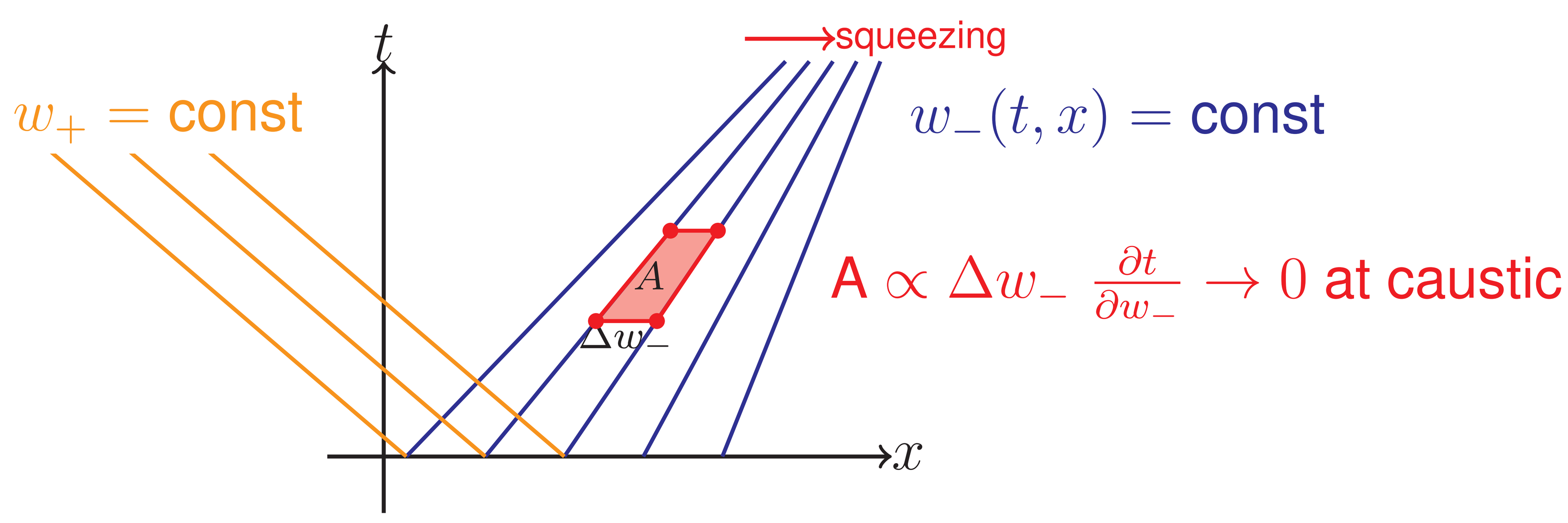
- Vacuum bias, $\epsilon \neq 0$: solves the cosmological DW problem.

Thin–DW model: Less energy loss / Less GW

- Effective action for the position Z of a thin, planar DW in the $1+3$ bulk: Dirac Born Infeld (DBI) ($\equiv$ Nambu–Goto in static gauge) with flat metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1)$ $(a = 0, 1, 2)$

$$S_{\epsilon DBI} = \int dy^0\, dy^1\, dy^2\left[-\sigma\sqrt{1 + \eta^{ab}\partial_a Z\partial_b Z} + \epsilon\ Z\right] \quad (1)$$

- Non-parallel cones of influence in DBI (Its characteristics)



- **Question:** Do characteristics intersect for **thin**–DW?

No. No caustics, no Shock Waves from smooth initial data

Proof in $1+1$**:** In DBI, Eqn. (1), follows a conservation law: for $Q_{\epsilon DBI} = \mathrm{e}^{\left(\epsilon\ \int_0^s f(s')ds'\right)}\ \frac{\partial t}{\partial w_-}$, (with $|f(s)| < \infty$)

$$\frac{\partial}{\partial s}\left[Q_{\epsilon DBI}\right] = 0\ . \quad (2)$$

Then $Q_{\epsilon DBI}$ = constant. Then $A \propto \frac{\partial t}{\partial w_-} \neq 0$

Proof in higher dimensions → See Weak discontinuities.

Thick–DW models

- Energy of a DW is localized over lengths, $\delta_{width} \approx \frac{\sqrt{2}}{\sqrt{\lambda}\ \langle\Psi\rangle}$
- $\mathcal{R}$: typical radius of curvature of the DW network.
- Corrections to the Thin DW as an expansion on $\frac{\delta_{width}}{\mathcal{R}} \ll 1$ (Curvature corrections: Polyakov, 1985; Maeda and Turok, 1987; Gregory (1988, 1990); Garfinkle and Gregory, 1989; Turok, 1989; Carter, 1994; Carter and Gregory, 1994; Anderson, Bonjour, Gregory and Stewart, 1997).
- A generic effective action for Thick $1+2$ DW (Blanco–Pillado, Martin–Caro, Jimenez–Aguilar and Queiruga, 2025),

$$L = P + G_3\,\Box Z + G_{4,X}\ \left((\Box Z)^2 - \partial_a\partial_b Z\ \partial^a\partial^b Z\right)\ , \quad (3)$$

with $P = -\sigma\sqrt{1 - 2\,X}$, $G_3 \propto \log\left(\sqrt{1 - 2\,X}\right)$. $X = -\frac{1}{2}\partial_a Z\partial^a Z$. Coupling constants fixed by field theory model.

Thick–DW can decay via Shock Waves

Theory (3) can develop shock–waves (Known from Horndeski theory Tanahashi and Ohashi, 2017.). Consider:

- Σ: the outermost edge of a localized wave on the $1+2$ DW.
- Σ at $\omega = 0$ contains the wavefront. $n^a = \frac{\partial^a\omega}{\sqrt{\partial_b\omega\partial^b\omega}}$ its normal.
- Localized wave as a discontinuity in $\partial_n\partial_n Z = n^a n^b\ \partial_a\partial_b Z$,

$$\Pi(\lambda) = [[\partial_n\partial_n Z]] = \partial_n\partial_n Z\Big|_{\Sigma^+} - \partial_n\partial_n Z\Big|_{\Sigma^-} \neq 0\ , \quad (4)$$

- Ricatti equation for the weak discontinuity Π from Eqn. (3),

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Pi + N(Z)\ \Pi^2 + M(Z)\ \Pi\Big|_{\Sigma} = 0\ . \quad (5)$$

Solution with initial value Π_0, and $Q = \int_0^\lambda d\lambda'\ M$

$$\Pi(\lambda) = \frac{-\Pi_0\ \exp\left(-Q\right)}{1 - \Pi_0\ \int_0^\lambda d\lambda'\ N\ \exp\left(-Q\right)} \quad (6)$$

Gradient Catastrophes on DW: Π can blow up in finite "time", if $(1 - \Pi_0\ \int_0^\lambda d\lambda'\ N\ \exp(-Q))$ vanishes. A shock wave. A relativistic wavefront is prone to shocks: $\partial_b\omega\partial^b\omega \to 0$ then $M \sim (\partial_b\omega\partial^b\omega)^{-2} \to \infty$; thus, enhancing the blow–up in (6).

Thin–planar DW remains exceptional: if $G_3 = G_{4,X} = 0$, no shock waves

$$N \propto \frac{-3\ P_{,XX} + P_{,X}\ P_{,XXX}}{P_{,X}}\Big|_{DBI} = 0.$$

(Exceptional DBI: Boillat, 1969; Deser, McCarthy and Sarioglu, 1998; [1])

Conclusions

1. **Thin** planar DW are exceptional (*nongenuinely nonlinear*), smooth, developing no caustics even with a vacuum bias [1].
   Thin wall approximation gives a critically incomplete model.
2. Realistic, **thick** Domain Walls develop gradient catastrophes. Shock Waves are **enhanced** in the relativistic limit of the wall.
3. Shock Waves could be a real mechanism of energy loss for **thick**–DW networks. Could be a reason for the *scaling regime*. Shock Waves could be a strong source SGWB.

[1] E. Babichev, B. Gafarov, S. Ramazanov, and M. Valencia-Villegas, *Caustic formation in DBI models: Wave propagation on planar domain walls*, Phys. Rev. D **113**, 116037 (2026) doi:10.1103/km7z-zhsk [arXiv:2604.08142 [hep-th]].

MORE INFORMATION: M. Valencia-Villegas
ITMP, MSU
mvalenciavillegas@itmp.msu.ru