CONFORMAL GAUGE GRAVITY AND LINEARIZED PARTICLE SPECTRA

Joosep Lember,[1,2] **Tomi Sebastian Koivisto**,[1,2] **Manuel Hohmann**[1]

[1]Institute of Physics, University of Tartu, Estonia
[2]National Institute of Chemical Physics and Biophysics, Tallinn, Estonia

Abstract

Shortcomings of general relativity (GR) motivate the study of extended theories of gravity, while other fundamental interactions, successfully formulated as gauge theories, drive the development of a gauge theory of gravity. We present a framework for gravity based on spontaneous symmetry breaking of the conformal group to the Lorentz group. The geometry is defined by a connection and two symmetry-breaking vector fields, leading to the construction of various classes of action densities. Applications of the framework are illustrated by the study of linearized theories around a Minkowski background, whose particle spectra are calculated using the *Wolfram Mathematica* package `PSALTer`. We find models with propagating degrees of freedom equivalent to those of general relativity as well as models with additional degrees of freedom.

Fundamental fields and spontaneous symmetry breaking

The gauge groups under consideration will be the conformal group $C(1,3) \cong SO(2,4)$ and $SO(1,5)$. The fundamental fields are **two orthogonal vectors** Φ^A **and** Ψ^A that transform under representations of the gauge group. In the case of $SO(1,5)$ the vectors can be constructed from an underlying 8-column spinor field ψ. In addition to the vector fields we have a Cartan **connection one-form** $\boldsymbol{\omega}^{AB} = \omega^{AB}{}_\mu \, \mathrm{d}x^\mu$ as a fundamental field.

The **spontaneous symmetry breaking mechanism** is performed by the two orthogonal vector fields, which take a specific symmetry broken form (symmetry breaking indicated with $\doteq$). One can also utilize another basis, where we have two null vectors

$$\Phi^A_- = \Phi^A + \Psi^A \doteq (0,0,0,0,\phi_-,\phi_-)\,, \qquad \Phi^A_+ = \Phi^A - \Psi^A \doteq (0,0,0,0,\phi_+,-\phi_+)\,. \tag{1}$$

The symmetry is broken in the two extra dimensions leaving either $SO(1,3)$ or $SO(4)$ symmetry in the cases of a $SO(2,4)$ or $SO(1,5)$ gauge group respectively.

The various different indices used are summarized in the table below:

Type of index	Notation	Possible values
Full gauge group $SO(2,4)$ or $SO(1,5)$	$A,B,C,\ldots$	0,1,2,3,4,5
Lorentz group $SO(1,3)$ or $SO(4)$	$a,b,c,\ldots$	0,1,2,3
Spacetime indices	$\mu,\nu,\rho,\ldots$	Depends on coordinates, e.g. in Cartesian t,x,y,z

Geometry

In order to describe the geometry of the spacetime manifold we need to define **tetrads**

$$\mathbf{e}^A = \frac{1}{2}\mathbf{D}\Phi^A_- \doteq (\boldsymbol{\theta}^a, \frac{1}{2}(\mathbf{d}+\boldsymbol{\omega}), \frac{1}{2}(\mathbf{d}+\boldsymbol{\omega}))\phi_-\,, \tag{2a}$$
$$\mathbf{f}^A = \frac{1}{2}\mathbf{D}\Phi^A_+ \doteq (\boldsymbol{\theta}^a, \frac{1}{2}(\mathbf{d}-\boldsymbol{\omega}), -\frac{1}{2}(\mathbf{d}-\boldsymbol{\omega}))\phi_+\,, \tag{2b}$$

where $\boldsymbol{\theta}^a = \theta^a{}_\mu \, \mathrm{d}x^\mu$, $\boldsymbol{\vartheta}^a = \vartheta^a{}_\mu \, \mathrm{d}x^\mu$, $\boldsymbol{\omega} = \omega^4{}_{5\mu} \, \mathrm{d}x^\mu \equiv W_\mu \, \mathrm{d}x^\mu$ and

$$\theta^a{}_\mu = \frac{1}{2}(\omega^a{}_{4\mu} + \omega^a{}_{5\mu})\,, \qquad \vartheta^a{}_\mu = \frac{1}{2}(\omega^a{}_{4\mu} - \omega^a{}_{5\mu})\,. \tag{3}$$

From the two tetrads we can define two **metrics**

$$g_{\mu\nu} = e^A{}_\mu e^B{}_\nu \eta_{AB}\,, \qquad f_{\mu\nu} = f^A{}_\mu f^B{}_\nu \eta_{AB}\,, \tag{4}$$

where $\eta_{AB} = \mathrm{diag}(-1,1,1,1,1,-1)$ for $SO(2,4)$ and $\eta_{AB} = \mathrm{diag}(1,1,1,1,1,-1)$ for $SO(1,5)$. Using the gauge covariant derivative $\mathbf{D}$, one can define the **conformal curvature**

$$\boldsymbol{F}^A{}_B = \mathbf{D}\boldsymbol{\omega}^A{}_B = \mathbf{d}\boldsymbol{\omega}^A{}_B + \boldsymbol{\omega}^A{}_C \wedge \boldsymbol{\omega}^C{}_B\,, \tag{5}$$

whose decomposition contains $\boldsymbol{R}^a{}_b \equiv \mathbf{d}\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b$ and $\boldsymbol{F}^4{}_5 = \mathbf{d}\boldsymbol{\omega} + 2\boldsymbol{\theta}^a \wedge \boldsymbol{\vartheta}_a \equiv \boldsymbol{F}$. Lastly, two **torsion tensors** are defined by

$$\boldsymbol{T}^A = \mathbf{D}\mathbf{e}^A \doteq (\boldsymbol{T}^a, \frac{\phi_-}{2}\boldsymbol{F}, \frac{\phi_-}{2}\boldsymbol{F})\,, \qquad \boldsymbol{S}^A = \mathbf{D}\mathbf{e}^A \doteq (\boldsymbol{S}^a, -\frac{\phi_+}{2}\boldsymbol{F}, \frac{\phi_+}{2}\boldsymbol{F})\,. \tag{6}$$

Action densities

Different action densities arise by constructing scalar invariants from the quantities introduced above. With two tetrads and torsion tensors available, similar Lagrangians may be formed using different combinations of these fields. We list below at least one representative example of each type.

- There are three Holst-type actions $\mathcal{L}_{H1} = \mathbf{e}^A \wedge \mathbf{e}^B \wedge \boldsymbol{F}_{AB}$.
- Three Palatini-type actions of the form $\mathcal{L}_{P1} = \epsilon_{ABCDEF}\Phi^E\Psi^F\mathbf{e}^A \wedge \mathbf{e}^B \wedge \boldsymbol{F}^{CD}$ and one of the form $\mathcal{L}_{P4} = \epsilon_{ABCDEF}\Phi^E\Psi^F\boldsymbol{F}^{AB} \wedge \boldsymbol{F}^{CD}$.
- We can define three torsion top forms $\mathcal{L}_{T1} = \boldsymbol{T}^A \wedge \boldsymbol{T}_A$
- and 16 different Weyl Curvature terms by combining Φ^A or Ψ^A with T^A or S^A, which in the symmetry-broken form become $\mathcal{L}_W \doteq \frac{\phi_-\phi_+}{4}\boldsymbol{F} \wedge \boldsymbol{F}$.
- We construct 8 scalar-torsion densities
$$\begin{aligned} &\mathcal{L}_a = \Phi^A_+\mathbf{e}_A \wedge \mathbf{e}^B \wedge \boldsymbol{T}_B\,, \quad \mathcal{L}_b = \Phi^B_-\mathbf{f}_B \wedge \mathbf{e}^A \wedge \boldsymbol{T}_A\,, \quad \mathcal{L}_c = \Phi^A_+\mathbf{e}_A \wedge \mathbf{f}^B \wedge \boldsymbol{T}_B\,, \\ &\mathcal{L}_d = \Phi^A_-\mathbf{f}_A \wedge \mathbf{f}^B \wedge \boldsymbol{T}_B\,, \quad \mathcal{L}_e = \Phi^A_+\mathbf{e}_A \wedge \mathbf{e}^B \wedge \boldsymbol{S}_B\,, \quad \mathcal{L}_f = \Phi^A_-\mathbf{f}_A \wedge \mathbf{e}^B \wedge \boldsymbol{S}_B\,, \\ &\mathcal{L}_g = \Phi^A_+\mathbf{e}_A \wedge \mathbf{f}^B \wedge \boldsymbol{S}_B\,, \quad \mathcal{L}_h = \Phi^A_-\mathbf{f}_A \wedge \mathbf{f}^B \wedge \boldsymbol{S}_B\,, \end{aligned} \tag{7}$$
which are used in the particle-spectrum analysis.
- Finally, there are 5 bimetric densities $\mathcal{L}_{B1} = \epsilon_{ABCDEF}\Phi^E\Psi^F\mathbf{e}^A \wedge \mathbf{e}^B \wedge \mathbf{e}^C \wedge \mathbf{e}^D$
- and an odd bimetric density $\mathcal{L}_{Odd} = \mathbf{e}^A \wedge \mathbf{e}^B \wedge \mathbf{f}_A \wedge \mathbf{f}_B$.

Symmetry breaking of the geometric quantities leads to the symmetry-broken form of the Lagrangians.

Calculation of background values

Prior to studying the linearized models one has to determine the background values of the fundamental fields that induce Minkowski background. We follow a similar procedure as presented in [1].

Minkowski spacetime is symmetric under the group $ISO(1,3)$, thus from [1], there has to exist a Lie group homomorphism $\tilde{\Lambda} : ISO(1,3) \times M \to SO(p,6-p)$, $p \in \{1,2\}$, such that the fields satisfy

$$\mathcal{L}_{X_\xi}\Phi^A_{(0)} = X^\mu_\xi \partial_\mu \Phi^A_{(0)} = -\lambda^A_\xi{}_B \Phi^B_{(0)}\,, \quad \mathcal{L}_{X_\xi}\Psi^A_{(0)} = X^\mu_\xi \partial_\mu \Psi^A_{(0)} = -\lambda^A_\xi{}_B \Psi^B_{(0)}\,, \tag{8a}$$
$$\mathcal{L}_{X_\xi}\omega^A_{(0)B\mu} = X^\nu_\xi \partial_\nu \omega^A_{(0)B\mu} + \partial_\mu X^\nu_\xi \omega^A_{(0)B\nu} = D_\mu \lambda^A_\xi{}_B\,, \tag{8b}$$

where X_ξ is the Killing vector field generating the symmetry $\xi \in \mathfrak{so}(p,6-p)$ and λ_ξ are the corresponding generators. In practice, we have to fix the homomorphism, that depends on the full gauge group, and a Cartesian coordinate system. Homomorphism is chosen such that

1. spacetime translations are mapped to the identity $\implies \lambda^A{}_B = 0$;
2. Lorentz transformations are mapped to the $SO(2,4)$ (or $SO(1,5)$) representations of $SO(1,3)$ (or $SO(4)$) $\implies$ only $\lambda^a{}_b \neq 0$.

Solving the symmetry conditions (8) for Killing vectors X^μ_ξ in Cartesian coordinates yields the background values for both gauge groups

$$\boxed{\Phi^{(0)A}_- = (0,0,0,0,\phi^{(0)}_-,\phi^{(0)}_-)\,, \quad \Phi^{(0)A}_+ = (0,0,0,0,\phi^{(0)}_+,-\phi^{(0)}_+)\,, \quad \omega^{ab}_{(0)\mu} = 0\,, \quad W^{(0)}_\mu = 0\,,} \tag{9}$$

where $\phi^{(0)}_\pm = const$. It can be seen that these induce Minkowski background as

$$\theta^a_{(0)\mu} = \delta^a{}_\mu = \vartheta^a_{(0)\mu} \implies e^A_{(0)\mu} \propto \delta^a{}_\mu\,, \ f^A_{(0)\mu} \propto \delta^a{}_\mu \implies g^{(0)}_{\mu\nu} \propto \eta_{\mu\nu}\,, \ f^{(0)}_{\mu\nu} \propto \eta_{\mu\nu}\,, \tag{10}$$

Particle spectra

`PSALTer`, developed by authors of [2, 3], calculates the propagating degrees of freedom (d.o.f.) of a Lagrangian expanded to 2nd order around a Minkowski background. The symmetry-broken form of the Lagrangian is written in terms of the components of the fundamental fields $\phi_\pm, \theta^a{}_\mu, \vartheta^a{}_\mu, W_\mu, \omega^{ab}{}_\mu$ and their linear perturbations. Since the geometry defined by `PSALTer` is strictly Lorentzian we are restricted to the case of $SO(2,4)$.

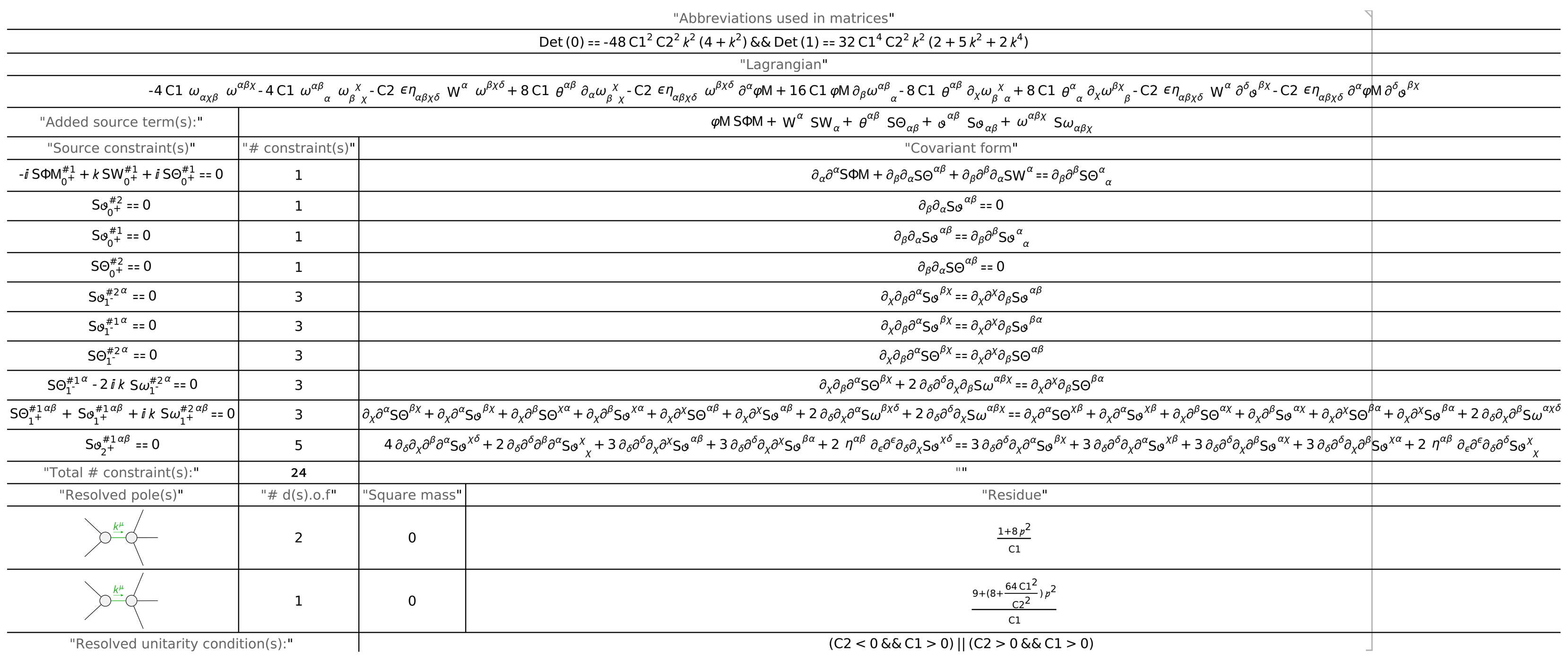
The goal is to study linear combinations of the gravitational actions with the Einstein-Hilbert terms, that in the symmetry-broken form are given by

$$\mathcal{L}^{GR}_e \doteq \epsilon_{abcd}\,\mathbf{e}^a \wedge \mathbf{e}^b \wedge \boldsymbol{R}^{cd}\,, \qquad \mathcal{L}^{GR}_f \doteq \epsilon_{abcd}\,\mathbf{f}^a \wedge \mathbf{f}^b \wedge \boldsymbol{R}^{cd}\,. \tag{11}$$

Here in particular, we report combinations of scalar-torsion actions (7) and the odd bimetric action $\mathcal{L}_{Odd}$ with the Einstein-Hilbert terms.

1. $\mathcal{L}^{GR}_e$ with any $\mathcal{L}_a$ to $\mathcal{L}_d$ OR $\mathcal{L}^{GR}_f$ with any $\mathcal{L}_e$ to $\mathcal{L}_h$ result in **equivalent degrees of freedom to those of GR**. The same happens for linear combinations of $\mathcal{L}^{GR}_e$ or $\mathcal{L}^{GR}_f$ with $\mathcal{L}_{Odd}$. The number of **gauge symmetries** changes compared to pure GR.
2. $\mathcal{L}^{GR}_e$ with any $\mathcal{L}_e$ to $\mathcal{L}_h$ OR $\mathcal{L}^{GR}_f$ with any $\mathcal{L}_a$ to $\mathcal{L}_d$ contain an **additional massless scalar degree of freedom**.

An example of the particle spectra of $C_1\mathcal{L}^{GR}_e + C_2\mathcal{L}_e$ can be seen below, where C_1 and C_2 are some constants. `PSALTer` identifies 24 gauge symmetries, each corresponding to a source constraint. At the bottom the propagating degrees of freedom are displayed. In the present example there are two massless d.o.f. equivalent to GR and an additional massless scalar d.o.f. Unitarity conditions ensure the absence of ghosts and tachyons via the positivity of the residue and thus $C_1 > 0$.



Other combinations of models could provide even richer particle spectra. Future works also include studying the models around a cosmological background.

Takeaway message

A gauge theoretic approach to gravity can be constructed from the spontaneous symmetry breaking of the conformal group. Studying linearized particle spectra of various actions indicates that models with additional degrees of freedom can be found amongst the ones with equivalent dynamics to GR.

References

[1]M. Hohmann, L. Järv, M. Krššák, and C. Pfeifer, "Modified teleparallel theories of gravity in symmetric spacetimes", Physical Review D **100**, 084002 (2019).

[2]W. Barker, C. Marzo, and C. Rigouzzo, "Psalter: particle spectrum for any tensor lagrangian", arXiv preprint arXiv:2406.09500 (2024).

[3]W. Barker, G. K. Karananas, and H. Tu, "Particle spectra of parity-violating theories: a less radical approach and an upgrade of the particle spectrum for any tensor lagrangian framework", Physical Review D **112**, 084041 (2025).