Thermodynamics of STU Solitons and Black Holes

Jack W. Lindner, Robert B. Mann

Introduction

The STU model originates from a Abelian truncation of a gauged $N = 8$ supergravity in four dimensions. This gives the theory a unique structure, its bosonic sector containing four $U(1)$ charges and three complex scalar fields [1]. The effective action is:

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{4}\sum_i \partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{4}\sum_{\Lambda=1}^{4} Y_\Lambda F^\Lambda_{\mu\nu}F^{\Lambda\mu\nu} + \frac{1}{L^2}\sum_i \cosh\phi_i\right) \tag{1}$$

From this action, one can derive equations of motions for the fields and the field equations that give rise to gravitational dynamics:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{(\phi)}_{\mu\nu} + T^{(F)}_{\mu\nu} \tag{2}$$

Black Hole and Soliton Thermodynamics

The thermodynamics of black holes has been researched for several decades, stretching back to the well-known results of Hawking, Page, and Bekenstein. Now, a rich study drawing familiar analogues to thermodynamic system has emerged and provides tools for investigating gravitational objects and phenomena in exotic theories of quantum gravity, supergravity, and alternatives to general relativity. Here we use the framework of black hole thermodynamics to investigate behavior in the STU model.

In our model, the metric for a black hole is given by:

$$ds_b^2 = -\frac{f_b(r)}{\sqrt{H_b(r)}}dt_b^2 + \frac{\sqrt{H_b(r)}}{f_b(r)}dr^2 + \frac{r^2}{L^2}\sqrt{H_b(r)}(dx^2 + dz_b^2) \tag{3}$$

where

$$f_b(r) = \frac{r^2}{L^2}H_b(r) - \frac{M}{r}, \quad H_b(r) = \prod_\Lambda\left(1 + \frac{Q_\Lambda^2}{Mr}\right)$$

The corresponding soliton solution is obtained via a double Wick rotation of the black hole metric ($dt_b \to idz_s$, $dz_b \to idt_s$) and analytic continuation of the charge ($Q \to -iQ$) [2]:

$$ds_s^2 = -\frac{r^2}{L^2}\sqrt{H_s(r)}dt_s^2 + \frac{\sqrt{H_s(r)}}{f_s(r)}dr^2 + \frac{r^2}{L^2}\sqrt{H_s(r)}dx^2 + \frac{f_s(r)}{\sqrt{H_s(r)}}dz_s^2 \tag{4}$$

By enforcing regularity of the metric to prevent conical defects, we obtain the period of the soliton (and the corresponding temperature for the black hole):

$$\eta_s = \frac{4\pi\sqrt{H_s(r_0)}}{f'(r_0)}, \quad f(r_0) = 0 \tag{5}$$

Additionally, from the model we obtain the magnetic flux of the soliton and the corresponding electric potential of the black hole:

$$\Phi_s = \frac{Q_{\Lambda s}}{\sqrt{2}r_0H_{\Lambda s}(r_0)}\eta_s, \quad \phi_b = \frac{Q_{\Lambda b}}{\sqrt{2}r_0H_{\Lambda b}(r_0)} \tag{6}$$

Results

Using the framework above, the relevant thermodynamic quantities were computed for both the soliton and the black hole numerically using Python.

In order to perform the analysis, the period of the soliton and black hole were set equal to a constant value:

$$\eta_s = \eta_b = C \tag{7}$$

With this fixed constraint, the value of Q_1 for the soliton was solved in terms of the other charges, L, and C, similar to analysis for other thermodynamic systems in non-linear electrodynamic (NLED) theories [3]. From this, the mass M, magnetic flux, and free energies were obtained for a variety of parameters.

For this poster, we will be focusing on the case where only Q_1 and Q_2 are non-zero, however the general case has been numerically computed, although is not yet fully understood.

Through this analysis, we have obtained many new behaviors that show a rich thermodynamic structure present in the STU model, including:

- Multiple branches of valid solitons that form multi-critical points with the black hole.
- Branch structure that has different solutions that depend on charge parameters.
- Multi-critical behavior sourced differently from previously studied NLED theories.

By studying the curve of G_s vs. Φ_{s_1}, we see a rich structure emerge, even for the case of only two non-zero charges. Multiple transitions are seen between solitons and the black hole as the flux is varied. The free energy of the black hole is constant on this plot, and calculated for $Q_{1b} = 1$, with the other charges equal to those of the soliton.

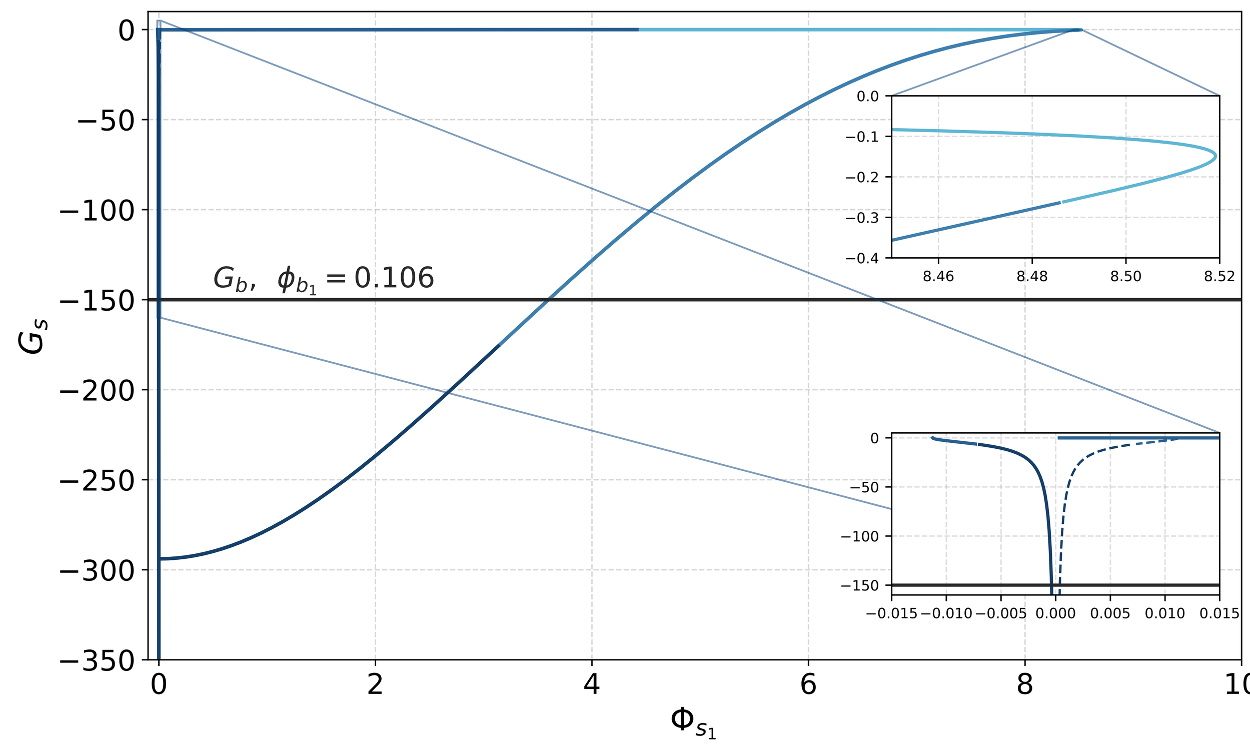

Figure 1. G_s vs. Φ_{s_1} for the parameters $Q_2 = 0.8$, $Q_3 = Q_4 = 0$, $L = 1$, $C = 0.5$. Different colours represent different solution branches. Dashed branches represent solutions belonging to the relevant negative Q_1 values.

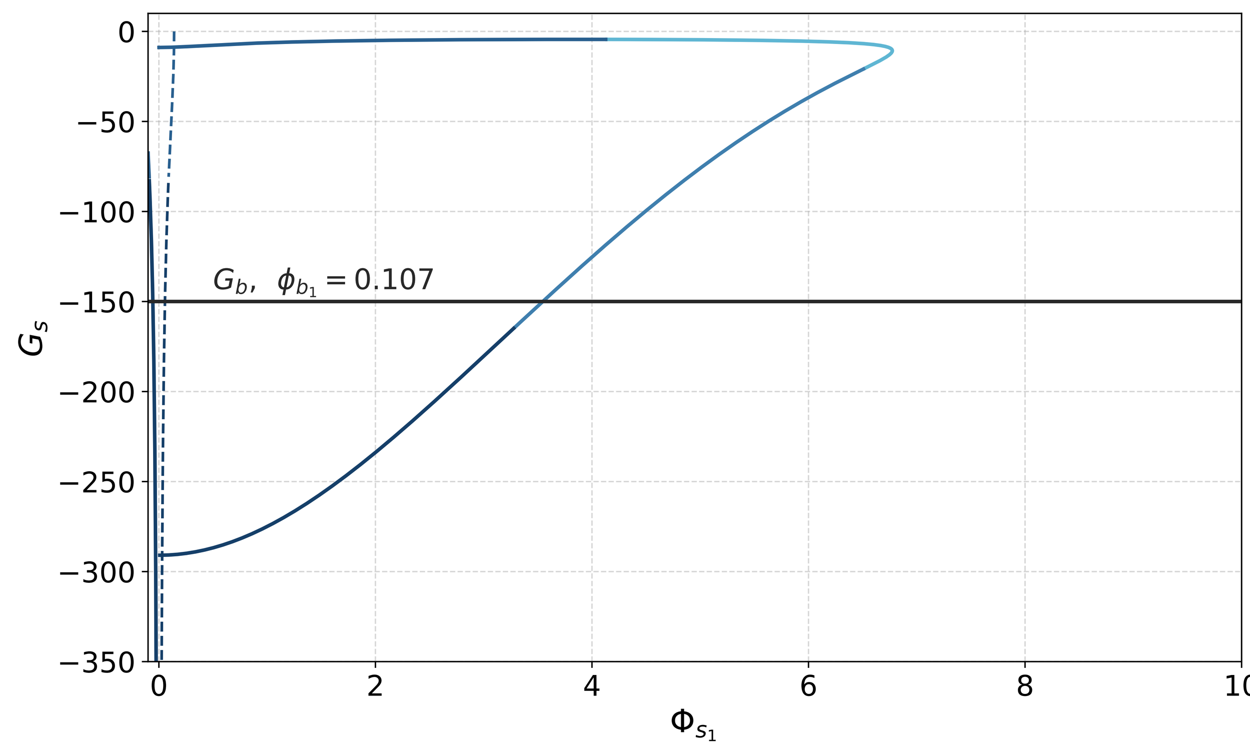

Figure 2. G_s vs. Φ_{s_1} for the parameters $Q_2 = 10$, $Q_3 = Q_4 = 0$, $L = 1$, $C = 0.5$.

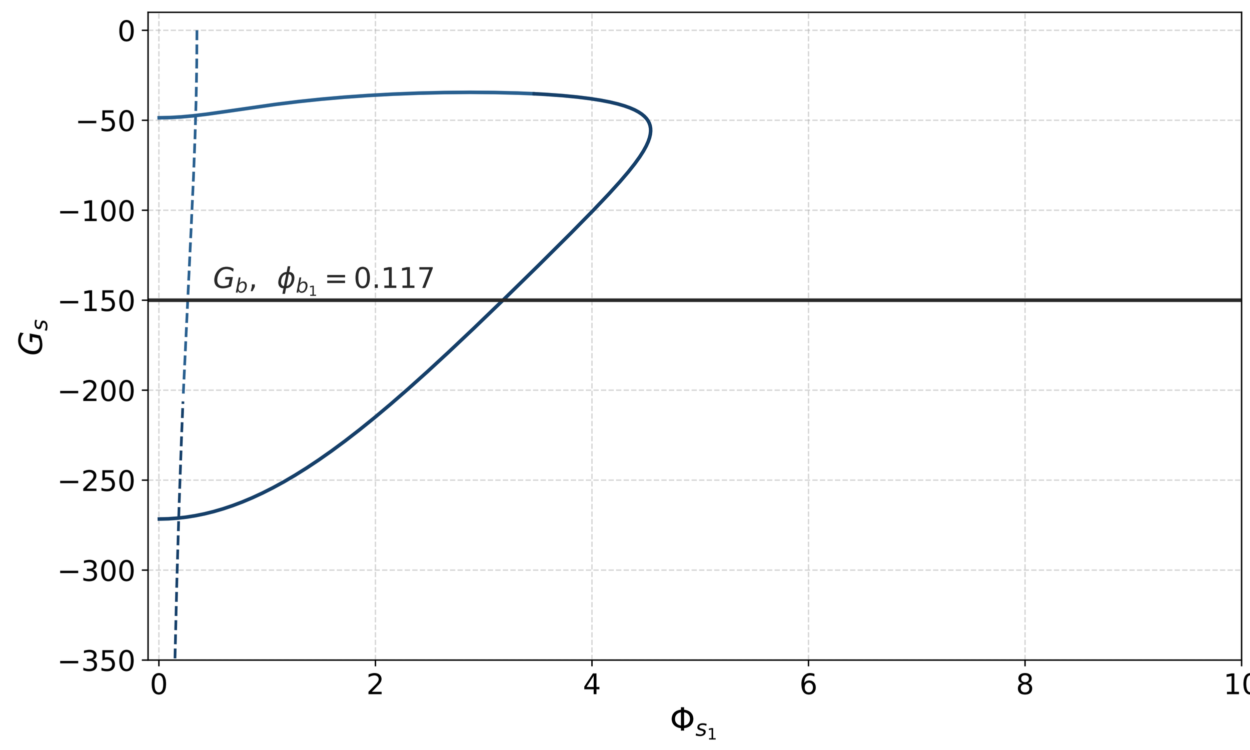

Figure 3. G_s vs. Φ_{s_1} for the parameters $Q_2 = 25$, $Q_3 = Q_4 = 0$, $L = 1$, $C = 0.5$.

Future Work

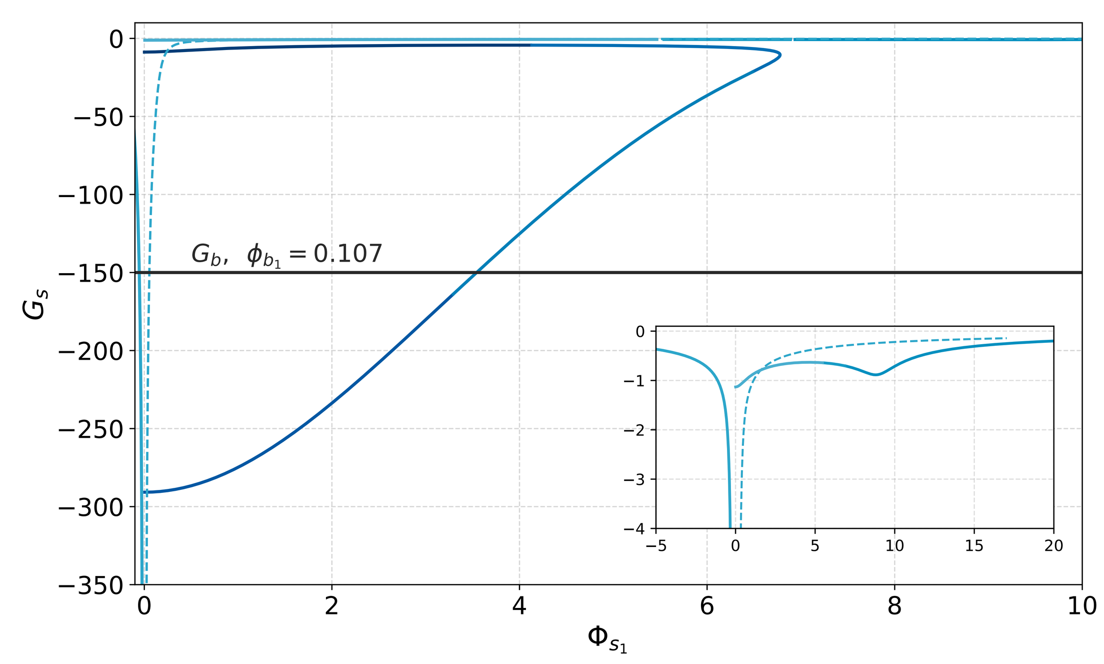

Figure 4. G_s vs. Φ_{s_1} for the parameters $Q_2 = 1$, $Q_3 = 10$, $Q_4 = 2$, $L = 1$, $C = 0.5$.

Future work for this study of STU thermodynamics includes:

- Understanding the effect of the three and four-charged cases and their impacts on the phase transitions.
- Extend analysis to heat capacity, pressure, and further variables [4].
- Understanding how this behavior relates to the holographic thermodynamics of this system using the AdS/CFT correspondence.

References

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[2] Sumati Surya, Kristin Schleich, and Donald M. Witt. Phase Transitions for Flat adS Black Holes. *Physical Review Letters*, 86(23):5231–5234, June 2001. arXiv:hep-th/0101134.

[3] Moaathe Belhaj Ahmed, Moisés Bravo-Gaete, Robert B. Mann, and Constanza Quijada. Multicritical points of gravitational solitons and a black hole in four dimensions. *Physical Review D*, 114(4):044037, August 2026. arXiv:2605.24783 [hep-th].