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Odd-parity perturbations in purely-metric Horndeski theories

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Context

- In GR, astrophysical Black Holes (BHs) are uniquely described by the Kerr solution
- Beyond GR, e.g. in **Scalar-Tensor (ST) theories**, new branches of scalarised BHs emerge
- 🎯 Characterise those new solutions of tractable modified gravity and provide testable benchmarks regarding their GW signatures in the ringdown phase

Motivation

- Among all ST theories, **purely-metric ones** stand out: are purely based on curvature invariants, i.e. only metric d.o.f. propagate
- Theories solely based on curvature invariants naturally arise in the effective action of several QG proposals (string theory, asymptotic safety, dynamical triangulation)

Purely-metric Horndeski theories

🎯 Cubic non-shift symmetric Horndeski theories: well-behaved ST theories with second-order EoM:

$$S[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\left[P + Q\Box\phi + FR + 2F_X(L_1 - L_2) + GE_{\mu\nu}\phi^{\mu\nu} + \tfrac{1}{3}G_X(L_3 - 3L_4 + 2L_5)\right], \quad (1)$$

with

$$\phi_\mu = \nabla_\mu\phi, \quad \phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi, \quad X = \phi_\mu\phi^\mu \equiv \nabla_\mu\phi\nabla^\mu\phi,$$

and

$$L_1 = \phi_{\mu\nu}\phi^{\mu\nu}, \quad L_2 = (\Box\phi)^2, \quad L_3 = (\Box\phi)^3, \quad L_4 = \phi_{\mu\nu}\phi^{\mu\nu}\Box\phi, \quad L_5 = \phi_{\mu\nu}\phi^{\nu\rho}\phi^{\mu}_{\rho}. \quad (2)$$

🎯 Functions $P, Q, F, G \equiv P(\phi,X), Q(\phi,X), F(\phi,X), G(\phi,X)$ such that, e.g., $F_X = \partial F/\partial X$. To be equivalent to the ST action (6), the scalar EoM should not depend on X (pure constraint), i.e.

$$\begin{aligned} P(\phi,X) &= -V(\phi) + 2X^2(3 - \ln(X))\xi^{(4)}(\phi), & F(\phi,X) &= \phi - 2X(2 - \ln(X))\xi''(\phi), \\ Q(\phi,X) &= 2X(7 - 3\ln(X))\xi^{(3)}(\phi), & G(\phi,X) &= -4\ln(X)\xi'(\phi). \end{aligned} \quad (3)$$

On-shell equivalence of theories

To summarise: degenerate $f(R,\mathcal{G})$ theories (4)-(5) are on-shell equivalent ST theories of the form (6), which are themselves equivalent to cubic, non-shift symmetric Horndeski theories (1), provided that P, Q, F, G are written as in (3) [1].

Degenerate $f(R,\mathcal{G})$ theories

🎯 Generic $f(R,\mathcal{G})$ propagating only one dynamical scalar d.o.f. ϕ (to avoid instabilities):

$$S[g_{\mu\nu}] = \int d^4x\sqrt{-g}f(R,\mathcal{G}), \quad \mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \quad (4)$$

which satisfy the following degeneracy condition and constraint

$$\begin{aligned} f(R,\mathcal{G}) &= \phi R + \xi(\phi)\mathcal{G} - V(\phi), \\ 0 &= R + \xi'(\phi)\mathcal{G} - V'(\phi), \end{aligned} \quad (5)$$

expressed as a parametric solution with generic $\xi(\phi)$, $V(\phi)$.

Hence, degenerate $f(R,\mathcal{G})$ theories are on-shell equivalent to the following scalar-tensor theory:

$$S[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\left(\phi R + \xi(\phi)\mathcal{G} - V(\phi)\right). \quad (6)$$

🎯 Class of degenerate $f(R,\mathcal{G})$ that, as ST theories, admit a new branch of spontaneously "scalarised" BHs, i.e. non-local generalisation of Starobinsky inflation:

$$f(R,\mathcal{G}) = R + \frac{R^2}{6(\mu^2 - \alpha\mathcal{G})}. \quad (7)$$

Correspond to ST theories with

$$\phi = 1 + \frac{R}{3(\mu^2 - \alpha\mathcal{G})}, \quad \xi(\phi) = \frac{3}{2}\alpha(\phi - 1)^2, \quad V(\phi) = \frac{3}{2}\mu^2(\phi - 1)^2, \quad (8)$$

with $\mu^2_{\text{eff}} = \mu^2 - \alpha\mathcal{G}$ the spacetime-dependent squared effective mass of the propagating scalar d.o.f. ϕ. The latter is **purely-metric based, hence the name "curvaturon"**.

New solutions: curvaturised black holes ([2] and new)

🎯 Consider degenerate $f(R,\mathcal{G})$ of the form (7), in which there is a new branch of BH solutions and the Kerr solution is not always stable. Due to the equivalence with Horndeski theories, **these new BH solutions spontaneously "scalarise", or rather "curvaturise" under the dynamical growth of non-zero Ricci scalar**.

"Curvaturisation" of BHs occurs under a tachyonic instability

$$\mu^2_{\text{eff}} = \mu^2 - \alpha\mathcal{G} < 0, \quad \left(\Box - \mu^2_{\text{eff}}\right)\delta\phi = 0, \quad (9)$$

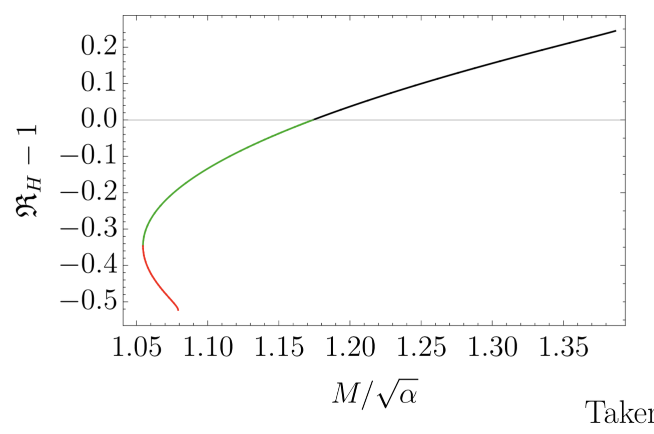
which develops around small-enough Schwarzschild BHs (solution of (7) for $\phi = 1$) under linearised scalar perturbations $\delta\phi$, provided that $\boldsymbol{M/\sqrt{\alpha} \leq 1.174}$, if $\alpha > 0$.

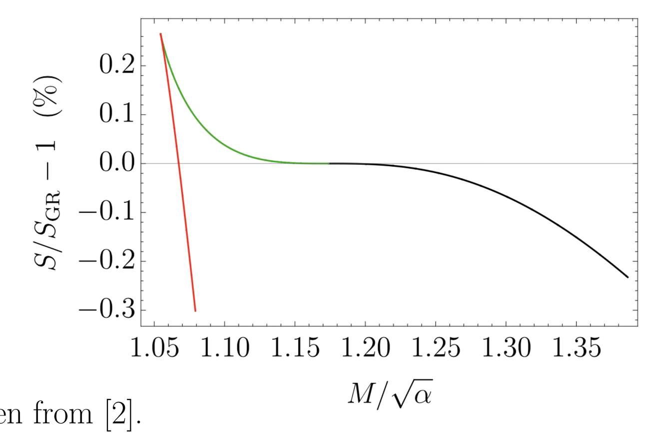
🎯 Curvaturised BHs are solutions of the EoM of the theory derived in the formulation (1) for a metric and scalar field of the form:

$$ds^2 = -f(r)e^{-2\delta(r)}dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2), \quad \phi \equiv \phi(r). \quad (10)$$

We obtain 3 coupled differential equations: 2 are first-order (for $f(r)$ and $\delta(r)$) and 1 is second-order (for $\phi(r)$). Those are **numerically solved using a shooting method for $\boldsymbol{\phi_H \equiv \phi(r_H)}$**, boundary conditions for $f(r)$, $\delta(r)$, $\phi(r)$ at r_H and r_∞, and reality + existence conditions.

New branch of "curvaturised" BHs branching out of the GR Schwarzschild solution for masses around $M/\sqrt{\alpha} \simeq 1.174$.





Taken from [2].

Axial perturbations: theory and results

🎯 Consider linearised ($\varepsilon \ll 1$) axial (odd-parity) perturbations around a background "curvaturised" BH $\{\bar{g}_{\mu\nu}(r), \bar{\phi}(r)\}$ of the form (10):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon\, h_{\mu\nu}, \quad \phi = \bar{\phi} + \varepsilon\delta\phi. \quad (11)$$

Expand the action up to the quadratic order $S^{(2)}[h_{\mu\nu},\phi] \propto \{\delta^{(2)}\phi,\, h^{(2)}_{\mu\nu},\, \delta\phi\cdot h_{\mu\nu}\}$ and restrict to linearised perturbation, i.e. terms $\propto \{\delta^{(2)}\phi,\, h^{(2)}_{\mu\nu}\} \to 0$. Compute the EoM for the perturbations $\{h_{\mu\nu}, \delta\phi\}$ by taking the variation

$$\mathcal{E}_{\mu\nu} = \frac{\delta S^{(2)}[h_{\mu\nu},\delta\phi]}{\delta h^{\mu\nu}} = 0, \quad \delta\phi \to 0, \quad (12)$$

as the scalar d.o.f. does not contribute to axial perturbations.

Use the Regge-Wheeler gauge for $h_{\mu\nu}$, i.e. only $\{h_{t\theta},\, h_{t\varphi},\, h_{r\theta},\, h_{r\varphi}\}$ remain, which depend on $h_0(t,r)$ and $h_1(t,r)$, and decompose in spherical harmonics $Y_{l,m}(\theta,\varphi)$ for fixed l, m. We obtain 3 (radial) EoM $\{\mathcal{E}_{t\theta},\, \mathcal{E}_{r\theta},\, \mathcal{E}_{\theta\theta}\} = 0$ and show that these and their derivatives up to 3rd are uniquely related, thus only 2 of them remain:

$$\mathcal{E}_{r\theta}(\omega, h_0, h_1, h_0'| \ldots) = 0, \quad \mathcal{E}_{\theta\theta}(\omega, h_0, h_1, h_0', h_1'| \ldots) = 0 \quad (13)$$

🎯 Motivated by [3], we rewrite those equations as

$$\frac{dY}{dr} = MY, \quad Y = \begin{pmatrix} h_0 \\ h_1/\omega \end{pmatrix}, \quad M = \begin{pmatrix} 2/r & -i\omega^2 + 2i\lambda\Phi/r^2 \\ -i\Gamma & \Delta \end{pmatrix}, \quad \lambda = \frac{l(l+1)}{2} - 1, \quad (14)$$

and show that those axial perturbations can be rewritten as the GR ones in the following effective metric

$$d\tilde{s}^2 = \Xi\left[-\Phi dt^2 + \Gamma\Phi dr^2 + r^2 d\Omega^2\right], \quad \Phi,\, \Gamma,\, \Delta,\, \Xi = \text{Func}\left(f(r), \delta(r), \phi(r), X, F(\phi,X), G(\phi,X)\right). \quad (15)$$

Next steps

- Are causality structures of background and perturbed spacetime compatible?
- Are curvaturised BHs stable?
- How do the axial Quasi-Normal Modes compare to GR?

References

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