Improved Quantum Null Energy Condition

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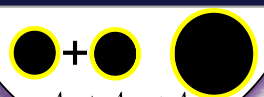
Black Hole Area Theorem

Discovered by Stephen Hawking in 1971

$$\Delta A \geq 0$$

The total surface area of the black hole's event horizon can never decrease over time.



$A_1 + A_2 < A$

Second Law of Thermodynamics

Total entropy must always increase!

$$\Delta S_{\text{universe}} \geq 0$$

What happens to entropy when matter falls into a black hole?

Entropy disappears?

Violation!

THERMO / GR

THERMO $\Delta S_{\text{universe}} \geq 0$

GR $\Delta A \geq 0$

What if the expanding area is the missing entropy?

In 1972, Jacob Bekenstein proposed: The black hole itself must carry entropy!

$$A_{BH} \propto S_{BH}$$

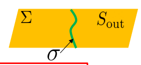
Generalized Second law

The area increase of the black hole will compensate or overcompensate for the lost matter entropy.

$S_{\text{gen}} \equiv S_{BH} + S_{\text{matter}}$ $\quad \mathrm{d}S_{\text{gen}} \geq 0$

Extending the notion of generalized entropy

Cauchy Surface (Σ)
Codimension-2 Surface (σ)



$$S_{\text{gen}} = \frac{A}{4G\hbar} + S_{\text{out}}$$

Classical Picture

$$\theta = \frac{1}{\delta A}\frac{\mathrm{d}(\delta A)}{\mathrm{d}\lambda}$$

Classical Expansion

$$T_{kk} \geq 0$$

Null Energy Condition

Classical Focusing Theorem: It states that if the null energy condition holds, gravity causes initially converging light rays to focus together and develop a caustic or singularity within finite affine parameter.

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \leq 0$$

Quantum Picture

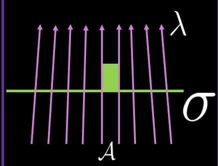


Fig: Purple line shows the null geodesics and surface σ is deformed along the affine parameter λ.

Quantum Expansion

$$\Theta = \frac{4G\hbar}{\delta A}\frac{\mathrm{d}S_{\text{gen}}}{\mathrm{d}\lambda}$$

Quantum Focusing Conjecture: It states that quantum expansion of light rays cannot increase along a null congruence.

Quantum Null Energy Condition

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} \leq 0 \xrightarrow{\theta = 0} T_{kk} \geq \lim_{\mathcal{A}\to 0}\frac{\hbar}{2\pi\mathcal{A}}\frac{\mathrm{d}^2 S_{\text{out}}}{\mathrm{d}\lambda^2}$$

$\downarrow \Theta = 0$

Improved Quantum Null Energy Condition

$$T_{kk} \geq \lim_{\mathcal{A}\to 0}\frac{\hbar}{2\pi\mathcal{A}}\left(\frac{\mathrm{d}^2 S_{\text{out}}}{\mathrm{d}\lambda^2} - \frac{d-3}{d-2}\theta S'_{\text{out}}\right)$$

Modular Hamiltonian

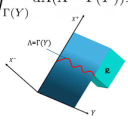
$$K_A^{\Psi} = -\ln \rho_A^{\Psi}$$

ρ is the reduced density matrix for the region A

In some cases, modular Hamiltonian can be written in terms of stress energy tensor.

$$K = 2\pi \int_{\mathbb{R}^{d-2}} \mathrm{d}^{d-2}Y \int_{\Gamma(Y)}^{\infty} \mathrm{d}\Lambda(\Lambda - \Gamma(Y))T_{\Lambda\Lambda}(\Lambda, Y)$$

Half Space



Quantum Relative Entropy

Measure of distinguishability between two quantum states

$$S_{\text{rel}}(\rho_{\mathcal{R}}||\sigma_{\mathcal{R}}) = \mathrm{Tr}(\rho \ln \rho) - \mathrm{Tr}(\rho \ln \sigma)$$

Convexity of Relative entropy

$$\frac{\mathrm{d}^2 S_{\text{rel}}(\rho_{\mathcal{R}}||\sigma_{\mathcal{R}})}{\mathrm{d}\lambda^2} \geq 0$$

Relation with modular Hamiltonian

$$S_{\text{out}} = \langle K \rangle - S_{\text{rel}}$$

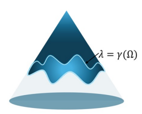
Proof

Consider the past light cone

Variation of the cut gives the variation of the generalized entropy.

And then using the convexity property of relative entropy, we get our required result.

$$K = 2\pi \int d\Omega_{d-2} \int_0^{\gamma(\Omega)} d\lambda\, \lambda^{d-1}\frac{\gamma(\Omega) - \lambda}{\gamma(\Omega)} T_{\lambda\lambda}$$



Conclusion

- INEC is a statement relating the local energy density and shape derivative of the von Neumann entropy in the Minkowski space.
- Stronger condition than QNEC, includes the possibility of expanding or contracting geodesics.
- Using the properties of relative entropy and modular Hamiltonian associated with the null deformation of the sphere, we show that INEC holds under additional assumption.

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