Partition function of a volume of space in a higher curvature theory

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For a fixed, finite volume of a region with the topology of a ball, the leading saddle point partition function is the exponential of the Bekenstein–Hawking or the Wald entropy of its boundary:

$$\mathcal{Z}(\Lambda, V, m^2) = e^{-I_E} = e^{A_V/4G_{\text{eff}}} = e^{S_W}$$

with $\dfrac{1}{G_{\text{eff}}} = \dfrac{1}{G_3}\sqrt{1+\dfrac{\Lambda}{m^2}}$

1 Why fix a volume?

The entropy of the gravitational field was originally assigned to black hole horizons; Gibbons and Hawking extended it to de Sitter horizons:

$$S_{\text{BH}} = \frac{k_B c^3}{4G\hbar} A_H,$$
$$S_{\text{GH}} = \frac{k_B c^3}{4G\hbar} A_{\text{dS}} = \frac{3\pi k_B c^3}{G\hbar\Lambda}.$$

We follow the interpretation in which de Sitter entropy is the logarithm of the dimension of the Hilbert space. De Sitter space has no boundary and its total Hamiltonian vanishes identically, so

$$\dim \mathcal{H} = \operatorname{Tr}\mathbb{1} = \operatorname{Tr} e^{-\beta \hat{H}},$$

and the right hand side is the partition function, which can be evaluated in the Euclidean path integral formulation.

Jacobson and Visser defined the entropy of a spatial region of fixed proper volume with a boundary, in Einstein's theory with or without a cosmological constant. In the saddle point approximation the path integral under the fixed volume condition is dominated by *constrained instantons*, whose Euclidean action is minus the Bekenstein–Hawking entropy of the area bounding the volume.

There is one caveat: a mild singularity in the Ricci tensor and the energy–momentum tensor at the boundary, which does not make the action divergent. It is expected to be cured by higher powers of curvature — which motivates this computation.

2 The constrained partition function

The quantum gravity partition function with a constraint is

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}g\, e^{-I_E(g)}\\ &= \int \mathcal{D}\mu \int \mathcal{D}\xi \int \mathcal{D}g\, e^{-I_E(g)+\frac{1}{\hbar}\int \mathrm{d}\phi\, \xi(\phi)(C(g)-\mu)}\end{aligned}$$

with ξ the Lagrange multiplier and $C(g)-\mu = 0$ the constraint. Here $C(g) = \int \mathrm{d}^2x\sqrt{\gamma}$ and $\mu = V$, where the induced metric on the spatial disk is

$$\gamma_{\mu\nu} := g_{\mu\nu} - N^2(\partial_\mu\phi)(\partial_\nu\phi), \quad N := \left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right)^{-1/2}.$$

In $2+1$ dimensions this is really the area of a disk; we keep the word *volume* to conform with the higher dimensional cases.

3 The theory: Born–Infeld NMG

Einstein's gravity in $2+1$ dimensions has no local degrees of freedom. New massive gravity (NMG) has two propagating ones,

$$I_{\text{NMG}} = \frac{1}{\kappa^2}\int \mathrm{d}^3x\sqrt{-\det(g)} \times \left[\sigma R + \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right) - 2\lambda_0 m^2\right],$$

and it was extended in principle to infinite powers in the curvature expansion:

$$I_{\text{BI-NMG}} = -\frac{4m^2}{\kappa^2}\int \mathrm{d}^3x\left[\sqrt{-\det\!\left(g+\tfrac{\sigma}{m^2}G\right)} - \left(1-\tfrac{\lambda_0}{2}\right)\sqrt{-\det(g)}\,\right]$$

with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and $\sigma = \pm 1$. Its rather nice properties:

- For $\lambda_0 = 0$, and unlike any generic finite order theory besides Einstein's gravity with a cosmological constant, it has a *unique* maximally symmetric vacuum, and flat space is that vacuum. NMG, by contrast, has two.
- A unitary massive spin-2 degree of freedom (with ± 2 modes) of mass $m_g = m$ in flat space and $m_g = m\sqrt{1+\lambda}$ around AdS.
- Because the Lagrangian is a determinant, the theory is consistent at infinite order *or at any finite truncation*: it reproduces the extended NMG theories that are consistent with AdS/CFT duality and that have a c-function.

4 Field equations

The Euclidean action is written in the convenient form

$$I_E = -\frac{4m^2}{\kappa^2}\int \mathrm{d}^3x\sqrt{\det(g)}\; F(R,K,S),$$

$$F := \sqrt{1 - \frac{\sigma}{2m^2}\left(R + \frac{\sigma}{m^2}K - \frac{1}{12m^4}S\right)} - \left(1 - \frac{\lambda_0}{2}\right)$$

where the curvature invariants are

$$K := R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{2}R^2, \qquad S := 8R^{\mu\nu}R_{\mu\alpha}R^{\alpha}{}_{\nu} - 6RR_{\mu\nu}R^{\mu\nu} + R^3.$$

Varying the action together with the constraint gives

$$\mathcal{E}_{\mu\nu} := -\frac{1}{2}g_{\mu\nu}F + \frac{\partial F}{\partial R}\frac{\delta R}{\delta g^{\mu\nu}} + \frac{\partial F}{\partial K}\frac{\delta K}{\delta g^{\mu\nu}} + \frac{\partial F}{\partial S}\frac{\delta S}{\delta g^{\mu\nu}},$$

and the field equations become

$$\mathcal{E}_{\mu\nu} = \frac{\kappa^2}{8m^2\hbar}T_{\mu\nu}, \qquad T_{\mu\nu} := \frac{\xi(\phi)}{N}\gamma_{\mu\nu}$$

so the constraint acts as a source of perfect fluid *without an energy density but with pressure*. Their explicit form is cumbersome but known.

5 The saddle point metric

Take a metric with ϕ and θ as Killing coordinates,

$$\mathrm{d}s^2 := N^2(r)\,\mathrm{d}\phi^2 + h(r)\,\mathrm{d}r^2 + r^2\mathrm{d}\theta^2.$$

$N(r)$ and $h(r)$ are to be determined from the field equations and the boundary conditions. Plugging this metric into them gives differential equations that are still highly complicated, so, guided by Jacobson and Visser, we take the ansatz

$$h(r) := \frac{1}{1-\dfrac{r^2}{L^2}} = \frac{1}{1-\Lambda r^2}, \qquad L^2 = \frac{1}{\Lambda},$$

with L the de Sitter radius. The lapse is then fixed by two conditions:

i $N(R_V) = 0$: the lapse vanishes on the surface of the constraint volume, where R_V is found from $\int \mathrm{d}^2x\sqrt{\gamma} = V$.

ii $\mathrm{d}N/\mathrm{d}l\,\big|_{l=0} = 1$, with $l = R_V - r$ the distance from the constraint volume surface. This removes the singularity there.

6 $\lambda_0 = 0$ solution

For flat spacetime $\Lambda = 0$ and $h(r) = 1$. Requiring $\mathcal{E}_{\phi\phi} = 0$, since $T_{\phi\phi} = 0$, forces $N(r) = \alpha r^2 + \beta$; the two boundary conditions then give $-2\alpha R_V = 1$ and

$$N(r) = \frac{R_V^2 - r^2}{2R_V}, \qquad \int \mathrm{d}^2x\sqrt{\det(\gamma)} = \pi R_V^2 = V,$$

so $R_V = \sqrt{V/\pi}$. The Euclidean saddle metric is

$$\mathrm{d}s^2 = \frac{1}{4R_V^2}(R_V^2 - r^2)^2\mathrm{d}\phi^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2,$$

and the action of this constrained instanton is

$$I_E = -\frac{8\pi^2}{\kappa^2}R_V = -\frac{4\pi}{\kappa^2}A_V = -\frac{A_V}{4G_3}$$
$$A_V = 2\pi R_V, \qquad G_3 = \frac{\kappa^2}{16\pi}$$

the same as the Bekenstein–Hawking or Gibbons–Hawking results, except that here it is calculated not for a black hole or de Sitter horizon but for a bounded disk. Hence the dimension of the Hilbert space is $\mathcal{Z} = \exp(A_V/4G_3)$.

As a final step, for completeness, the Lagrange multiplier follows from

$$\mathcal{E}_{rr} = \frac{1}{m^2(R_V^2 - r^2)}, \qquad T_{rr} = \frac{2R_V}{R_V^2 - r^2}\xi,$$

which yields $\xi = 1/(4\pi G_3 R_V)$.

7 $\lambda_0 \neq 0$ solution

Set $r = L\sin\chi$ and $u = \sin\chi$. Assuming $N(\chi) = \alpha\cos\chi + \beta$, the first boundary condition gives $N = \alpha(\cos\chi - \cos\chi_V)$ and the second $\alpha = 1/\sin\chi_V$, so

$$N(\chi) = \frac{\cos\chi - \cos\chi_V}{\sin\chi_V}.$$

Now $\mathcal{E}_{\phi\phi} = \frac{\kappa^2}{8m^2\hbar}T_{\phi\phi} = 0$ fixes the relation between L^2, m^2 and the bare cosmological constant:

$$\frac{1}{L^2} = \Lambda = m^2\sigma\lambda_0\left(1 - \frac{\lambda_0}{4}\right).$$

The constraint gives the volume and the boundary area,

$$V = \int \mathrm{d}^2x\sqrt{\gamma} = 2\pi L^2(1-\cos\chi_V), \qquad A_V = 2\pi L\sin\chi_V,$$

and in terms of these the action becomes

$$I_E = \left(\frac{\pi}{G_3}\frac{V}{A_V}\right)\sigma\sqrt{1 - \frac{\sigma}{L^2m^2}}.$$

For a generic n-ball one has $V = \frac{R_V}{2}A_V = \frac{L\sin\chi_V}{2}A_V$, so this becomes

$$I_E = \frac{A_V}{4G_3}\sigma\sqrt{1 - \frac{\sigma\Lambda}{m^2}}$$

For $\Lambda = 0$ and $\sigma = -1$ this reduces to the flat space result above; from now on we set $\sigma = -1$.

8 The answer is the Wald entropy

For $\Lambda \neq 0$ this is exactly the same as the Wald entropy, defined as

$$S_W = -2\pi\oint \frac{\partial L}{\partial R_{\rho\sigma}}g^{\mu\nu}\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}\,\mathrm{d}^3x,$$

with $\varepsilon_{\mu\rho}$ the binormal to the constraint boundary surface. For BINMG in de Sitter spacetime this becomes

$$S_W = \frac{A}{4}\frac{1}{G_3}\sqrt{1 + \frac{\Lambda}{m^2}},$$

so the Euclidean action of the constrained instanton is exactly minus the Wald entropy, evaluated not at the de Sitter horizon but at the constraint boundary. Defining the effective gravitational constant

$$\frac{1}{G_{\text{eff}}} := \frac{1}{G_3}\frac{\bar{R}_{\mu\nu}}{\bar{R}}\frac{\partial L}{\partial R_{\mu\nu}}\bigg|_{\bar{R}} = \frac{1}{G_3}\sqrt{1 + \frac{\Lambda}{m^2}},$$

the Wald entropy is just like the Bekenstein–Hawking entropy, $S_W = A/4G_{\text{eff}}$, and

$$\mathcal{Z}(\Lambda, V, m^2) = e^{-I_E} = e^{\frac{A_V}{4G_3}\sqrt{1+\frac{\Lambda}{m^2}}} = e^{\frac{A_V}{4G_{\text{eff}}}}$$

As a final step, for completeness, the Lagrange multiplier is now $\xi = -\cot\chi_V/(4\pi G_3\sqrt{L^2m^2+1})$.

9 Conclusions

- The quantum gravity partition function was computed at the saddle point approximation in an infinite order gravity theory of the Born–Infeld form in $2+1$ dimensions — a theory with two propagating degrees of freedom, locally nontrivial unlike three dimensional Einstein gravity.
- The dimension of the Hilbert space of all states of a given disk with a boundary is equal to the exponential of the Bekenstein–Hawking, Gibbons–Hawking or Wald entropy of it. The computations support the conclusions of Jacobson and Visser.
- In the $\Lambda \neq 0$ case the effective Newton's constant appears in the entropy formula. The calculations lend support to the holographic nature of gravity for finite regions of space with a boundary.
- **Still open:** although the constrained instanton has a finite action, for both $\lambda_0 = 0$ and $\lambda_0 \neq 0$ there remains a mild singularity at the boundary for the Ricci tensor and the stress-energy tensor. It is *not* cured here.
- **Coming work:** a similar analysis in n-dimensional Born–Infeld gravity, which has the same perturbative and vacuum structure as Einstein's theory.

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