Static Charged or Uncharged Black Holes with External Bertotti-Robinson and Bonnor-Melvin Electromagnetic Fields in General Relativity and Supergravity

Andrea Di Pinto[a,b]

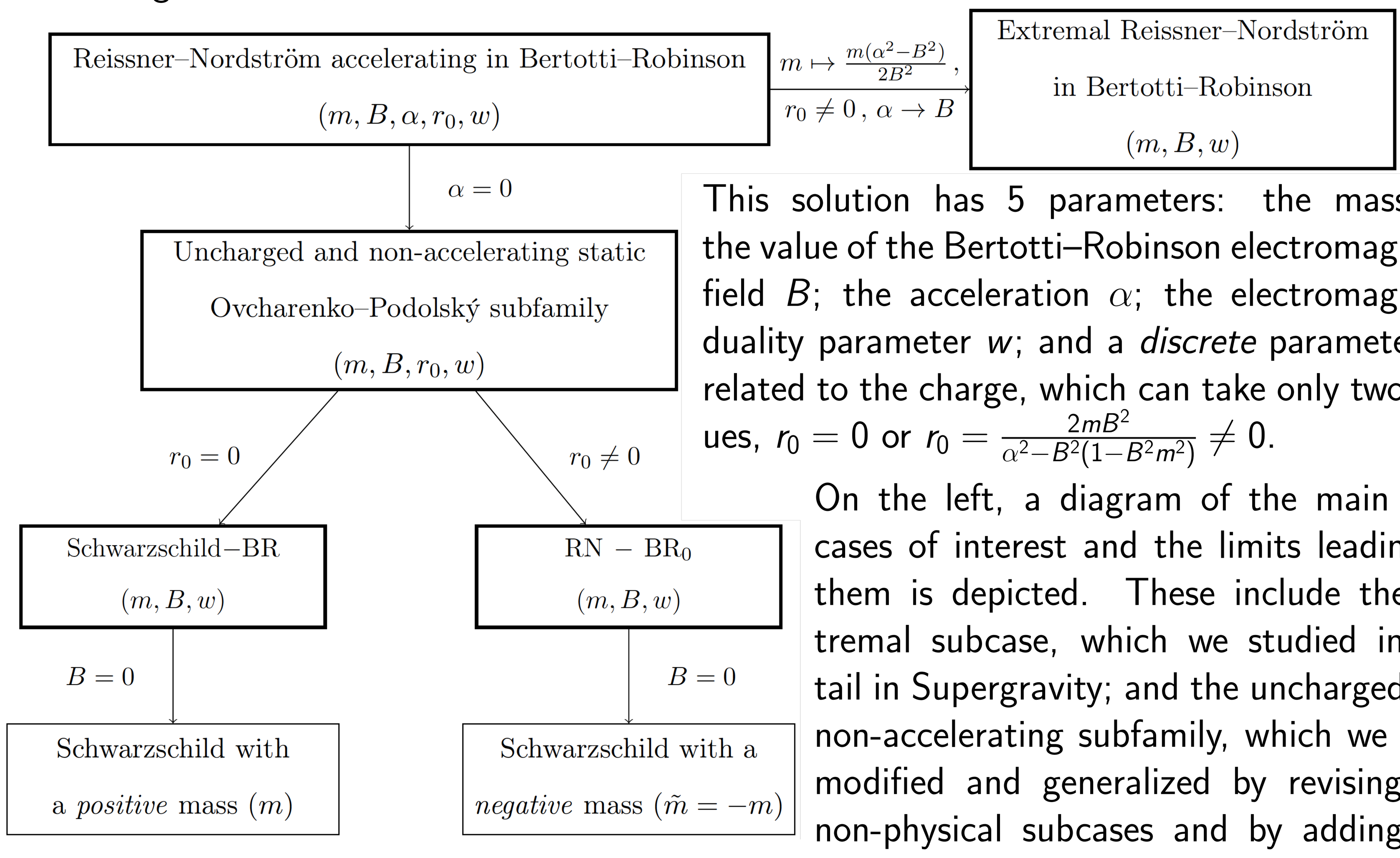
[a] Università degli Studi dell'Insubria, DISAT — Como, Italy

[b] Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Milano — Milan, Italy

Based on [1] and [2]

Starting Point: The Reissner–Nordström black hole in Bertotti–Robinson

In [3] it has been presented an exact solution of the Einstein–Maxwell field equations which describes a static charged Reissner–Nordström black hole accelerating in an external uniform electromagnetic field, called the Bertotti–Robinson universe.



This solution has 5 parameters: the mass m; the value of the Bertotti–Robinson electromagnetic field B; the acceleration α; the electromagnetic duality parameter w; and a *discrete* parameter r_0, related to the charge, which can take only two values, $r_0 = 0$ or $r_0 = \frac{2mB^2}{\alpha^2 - B^2(1-B^2m^2)} \neq 0$.

On the left, a diagram of the main subcases of interest and the limits leading to them is depicted. These include the extremal subcase, which we studied in detail in Supergravity; and the uncharged and non-accelerating subfamily, which we have modified and generalized by revising the non-physical subcases and by adding the Melvin electromagnetic field.

Supergravity of the Reissner-Nordström black hole in Bertotti–Robinson

We examine the supersymmetry of the general family of the static Reissner–Nordström black hole accelerating in Bertotti–Robinson, in the context of ungauged $\mathcal{N} = 2$, $D = 4$ supergravity, which is a theory characterized by four bosonic and four fermionic degrees of freedom: a graviton e^a_μ, a complex gravitino $\psi_\mu = \psi^1_\mu + \mathrm{i}\,\psi^2_\mu$, and a Maxwell gauge field A_μ.

An exact solution of the Einstein–Maxwell equations in general relativity can be viewed as representing a possible background of the bosonic sector of supergravity, $\psi_\mu = 0$. This can be achieved if the solution is supersymmetric, in the sense that there exists at least one Killing spinor ϵ which generates the local supersymmetry transformations and leaves the background invariant, $\delta\psi_\mu = \hat{\nabla}_\mu \epsilon = 0$, where $\hat{\nabla}_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{\mathrm{i}}{4}F_{ab}\gamma^{ab}\gamma_\mu$ is the supercovariant derivative.

However, instead of directly integrating this set of differential equations for the Killing spinors, one can first verify whether the necessary, but not sufficient, integrability conditions are satisfied, namely $det([\hat{\nabla}_\mu, \hat{\nabla}_\nu]) \equiv det(\hat{R}_{\mu\nu}) = 0$.

For this family of solutions we find that the integrability conditions are satisfied if and only if the black hole is extremal. For this subcase, solving the Killing equations results in *two* complex Killing spinors $\epsilon_{1,2}$, meaning that this solution is $\frac{1}{2}$–BPS and preserves half of the supersymmetries. Using the explicit expression of these Killing spinors, we also compute all the various bilinears that can be constructed from them: $\bar{\epsilon}\,\Gamma\,\epsilon$, with $\Gamma \in \{\mathbb{1}, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \gamma^{\mu\nu}\}$. In particular, the vectorial current $\mathrm{J}^\mu = \bar{\epsilon}\,\gamma^\mu\,\epsilon$ is found to be timelike $\mathrm{J}^\mu \mathrm{J}_\mu < 0$.

From Supergravity to Black Hole Thermodynamics

Computing the Komar mass for the extremal Reissner–Nordström black hole in Bertotti–Robinson results in a divergent quantity. However, since the spacetime is supersymmetric, it should saturate the BPS bound $\mathrm{M}^2 = \mathrm{Z}^2 = \mathrm{Q}^2 + \mathrm{P}^2 = m^2$, where Q is the black hole electric charge, while P is the magnetic one.

On the other hand, computing the black hole temperature T and entropy S gives the same result as the extremal Reissner–Nordström black hole without the Bertotti–Robinson field, i.e. $\mathrm{T} = 0$ and $\mathrm{S} = \pi m^2$.

As a result, the mass $\mathrm{M} = m$ defined above via the BPS bound satisfies the first law of black hole mechanics: $\delta\mathrm{M} = \mathrm{T}\,\delta\mathrm{S} + \Phi\,\delta\mathrm{Q}$, as well as the Smarr law: $\mathrm{M} = 2\,\mathrm{T}\,\mathrm{S} + \Phi\,\mathrm{Q}$ and the Christodoulou–Ruffini mass formula: $\mathrm{M}^2 = \frac{\mathrm{S}}{4\pi} + \frac{\mathrm{Z}^2}{2} + \frac{\pi\,\mathrm{Z}^4}{4\,\mathrm{S}}$, once the gauge constant in the electrostatic potential $\Phi = \Phi_{\mathrm{H}} + \Phi_{int} = \Phi_{int}$ is properly fixed by setting $\Phi_{int} = 1$.

Multi Black Hole Generalization and Adding a Positive Cosmological Constant

We prove that the extremal Reissner–Nordström black hole in Bertotti–Robinson is actually part of the Majumdar–Papapetrou class by providing the explicit change of coordinates, firstly in cylindrical Weyl coordinates (ρ, ϕ, z), and then in Cartesian ones $(x = \rho\cos\phi,\ y = \rho\sin\phi,\ z)$.

This result also allows one to obtain this spacetime and interpret it as the near-horizon limit of two extremal Reissner–Nordström black holes, one located at the origin with mass m, and the other located on the z–axis at a distance equal to its mass, $m_B = -z_B = 1/B$. This is consistent with the Bertotti–Robinson spacetime being the near-horizon limit of an extremal Reissner–Nordström black hole.

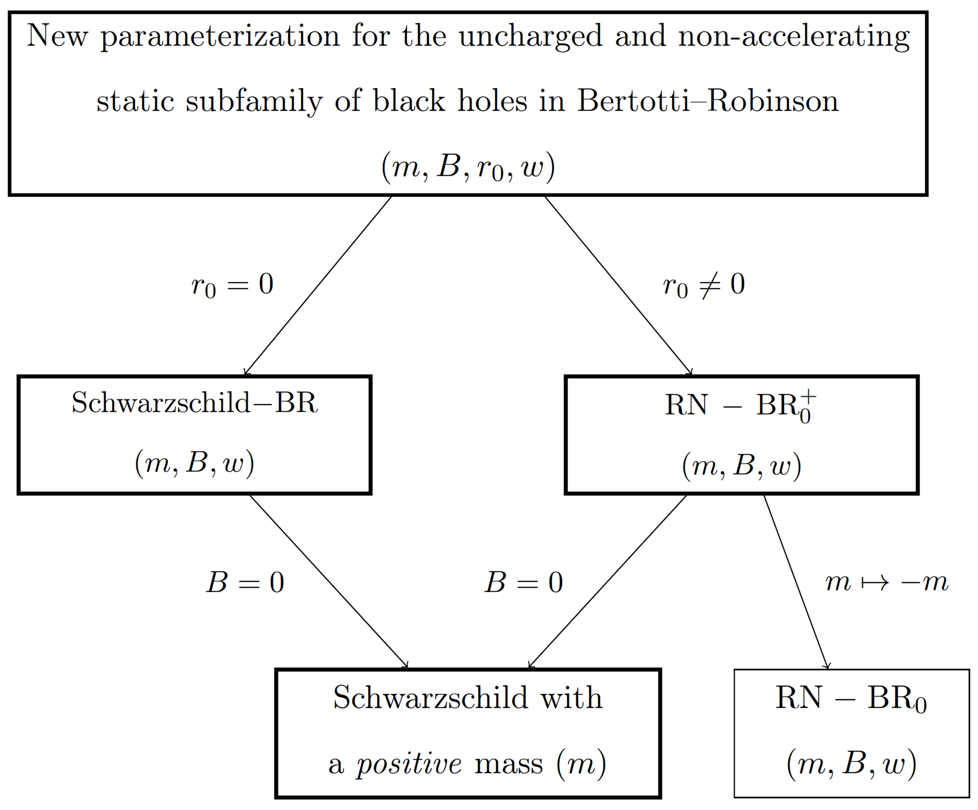
Moreover, this also permits the construction of a generalization with N generic sources and a *positive* cosmological constant $\Lambda > 0$. The resulting spacetime is given by:

$$\begin{aligned} ds^2 &= -U^{-2}dt^2 + U^2\left(dx^2 + dy^2 + dz^2\right)\,, \qquad U = \sqrt{\frac{\Lambda}{3}}\,t + \sum_{i=1}^{N}\frac{m_i}{|\vec{x} - \vec{x}_i|}\,, \\ A &= w\,U^{-1}dt + \sqrt{1-w^2}\sum_{i=1}^{N}\frac{m_i(z - z_i)}{|\vec{x} - \vec{x}_i|}\frac{\left[(x - x_i)dy - (y - y_i)dx\right]}{\left[(x - x_i)^2 + (y - y_i)^2\right]}\,, \end{aligned} \tag{1}$$

where the single extremal Reissner–Nordström black hole in Bertotti–Robinson is thus obtained by setting $N = 2$, $\vec{x}_1 = (0, 0, -1/B)$, $m_1 = 1/B$, $\vec{x}_2 = 0$, $m_2 = m$.

Fixing the Uncharged and Non-accelerating Subfamily: The General Schwarzschild Black Hole in Bertotti–Robinson

As depicted in the first diagram, the uncharged and non-accelerating subfamily of the Reissner–Nordström black hole in Bertotti–Robinson can be obtained simply by setting the acceleration to zero, $\alpha \to 0$. Nonetheless, while the $r_0 = 0$ branch of this subfamily reduces, in the $B \to 0$ limit, to a physical Schwarzschild black hole with positive mass, the $r_0 \neq 0$ branch reduces, in the same limit, to a Schwarzschild solution with negative mass.



Therefore, we provide a new parameterization of this subfamily, in which both branches reduce to the usual Schwarzschild solution with positive mass when the electromagnetic field is removed, $B \to 0$, as depicted in the diagram on the left.

Moreover, by introducing additional gauge constants, we remove the various pathologies, namely Dirac strings and conical singularities, and also ensure consistency with the laws of black hole thermodynamics. Surprisingly, proceeding in this way, one obtains the same results as for the Schwarzschild solution without the Bertotti–Robinson electromagnetic field: $\mathrm{M} = m$, $\mathrm{T} = \frac{1}{8\pi m}$, $\mathrm{S} = 4\pi m^2$. This spacetime is in principle described by the $b = 0$ subcase of equations (2); however, a more compact form can be found in [2].

Addition of the Melvin parameter: The General Schwarzschild Black Hole in Bertotti–Robinson–Bonnor–Melvin

We generalize this new uncharged and non-accelerating class of black holes in Bertotti–Robinson by also introducing the Melvin external electromagnetic field, as done in [4] for the $r_0 = 0$ branch. The addition of the Melvin parameter can be achieved by writing the seed solution in cylindrical Weyl coordinates (ρ, ϕ, z) in the so-called magnetic Lewis–Weyl–Papapetrou ansatz:

$$ds^2 = f^{-1}\left[-\rho^2 dt^2 + e^{2\gamma}\left(d\rho^2 + dz^2\right)\right] + f\left(d\phi - \omega\,dt\right)^2\,, \qquad A = A_t\,dt + A_\phi\,d\phi\,,$$

from which one can compute the complex Ernst potentials $\mathcal{E}$ and Φ as:

$$\begin{aligned} \mathcal{E} &= -f - |\Phi|^2 + ih\,, & \Phi &= \tilde{A}_t - iA_\phi\,, \\ \vec{e}_\phi \times \vec{\nabla}\tilde{A}_t &= -\rho^{-1}f\left(\vec{\nabla}A_t + \omega\vec{\nabla}A_\phi\right)\,, & \vec{e}_\phi \times \vec{\nabla}h &= -\rho^{-1}f^2\,\vec{\nabla}\omega - 2\,\vec{e}_\phi \times \mathrm{Im}\left(\Phi^*\vec{\nabla}\Phi\right)\,, \end{aligned}$$

and then generate a *new* solution of the Einstein–Maxwell equations through a so-called Harrison transformation: $\mathcal{E}' = \frac{\mathcal{E}}{1 - 2\,\alpha^*\Phi - \alpha^*\alpha\,\mathcal{E}}$, $\Phi' = \frac{\alpha\,\mathcal{E} + \Phi}{1 - 2\,\alpha^*\Phi - \alpha^*\alpha\,\mathcal{E}}$, where $\alpha = e + \mathrm{i}\,b$ is the transformation parameter, which therefore determines the strength of the added electromagnetic Melvin field.

The result of this procedure for the purely magnetic subcase $(w = 0, e = 0)$ is:

$$\begin{aligned} ds^2 &= -\frac{g^2\Delta_r}{r^2\Omega^2}\delta_t^2\,dt^2 + \frac{C_f^2\,g^2 r^2}{\Omega^2}\left[\frac{d\theta^2}{\Delta_\theta} + \frac{dr^2}{\Delta_r}\right] + \frac{r^2\sin^2\theta\,\Delta_\theta}{g^2}\delta_\phi^2\,d\phi^2\,, \\ A &= \left[B^{-2}g^{-1}\left[(B - 2b)(\Gamma - \Omega) + (b\,B^2 r_0^2)\Omega\right] - \delta A_\phi\right]\delta_\phi\,d\phi\,, \\ \Omega^2 &= \left[1 + B^2(r - r_0)^2\right] - \left[r^2(1 - B^2m^2) - 2mr\right]B^2\cos^2\theta\,, \qquad \Delta_\theta = 1 + B^2m^2\cos^2\theta\,, \\ \Delta_r &= \left[r^2(1 - B^2m^2) - 2mr\right]\left[1 + B^2(r - r_0)^2\right]\,, \qquad \Gamma = 1 + B^2\left[mr\cos^2\theta - r_0(r - r_0)\right]\,, \\ g &= B^{-2}\left[2b(B - b)\Gamma + \left[(2b^2 - 2bB + B^2) + b^2B^2r_0^2\right]\Omega\right]\,, \qquad r_0 = \{0,\ 2m(1 - B^2m^2)^{-1}\}\,, \end{aligned} \tag{2}$$

where δ_t, C_f, δ_ϕ, and δA_ϕ are the gauge constants that we introduce also in this more general spacetime in order to fix the pathologies and ensure that all formulas and laws of black hole thermodynamics are satisfied. In this way, one again obtains that the temperature T and entropy S are those of the Schwarzschild black hole without any external electromagnetic field, as above.

Vacuum Subcase(s)

We prove that, by setting the Melvin parameter to $b = \frac{B}{2 + B^2 m\,r_0}$ in (2), one obtains a vacuum solution (without an electromagnetic field, $A = 0$) which should generalize the solutions found in the same way in [5] for the $r_0 = 0$ branch, that represents a black hole in an external gravitational background, or in an expanding bubble of nothing. However, we formally prove that, when the electromagnetic field disappears, the r_0 parameter becomes inessential. Indeed, we provide the change of coordinates that removes it and motivate this result by the fact that this parameter controls whether the aligned and non-aligned parts of the electromagnetic field are independent of each other or not, thus becoming inessential when the electromagnetic field is removed.

References

[1] A. Di Pinto and A. Viganò, *Supersymmetry of the static Reissner-Nordström black hole in Bertotti-Robinson* $(\mathrm{AdS}_2 \times \mathrm{S}^2)$, [arXiv:2606.11101] [INSPIRE]

[2] A. Di Pinto, *A more general exact solution for the Schwarzschild black hole in a Bertotti-Robinson-Bonnor-Melvin universe*, [arXiv:2608.18213] [INSPIRE]

[3] H. Ovcharenko and J. Podolsky, *Static black holes in an external uniform electromagnetic field: Reissner-Nordström accelerating in Bertotti-Robinson*, Phys. Rev. D **114** (2026) no.2, 024068 [arXiv:2602.15462] [INSPIRE]

[4] M. Astorino, *Black holes in the external Bertotti-Robinson-Bonnor-Melvin electromagnetic field*, Phys. Rev. D **112** (2025) no.10, 104077 [arXiv:2508.12908] [INSPIRE]

[5] M. Astorino, *Static hairy black hole in 4D general relativity*, Phys. Rev. D **113** (2026) no.2, 024047 [arXiv:2601.16254] [INSPIRE]

andrea.dipinto@mi.infn.it